THE FINITE-DIFFERENCE TIME-DOMAIN (FDTD) METHOD – PART II

9. Some excitation functions (sources)

The problem of choosing the proper space-time dependence of the excitation $g(x,y,z,t)$ requires special attention. Here, we confine our attention to the time-dependence of the sources and their discretization.

Sinusoidal excitation

$$\sin(\omega t) = \sin\left(n\frac{2\pi}{T}\Delta t\right), \quad n = 0,1,2,3,\ldots$$

Assuming that $\Delta t = T/32 \quad \Rightarrow g^n = \sin\left(\frac{\pi}{16}n\right), \quad n = 0,1,2,\ldots$

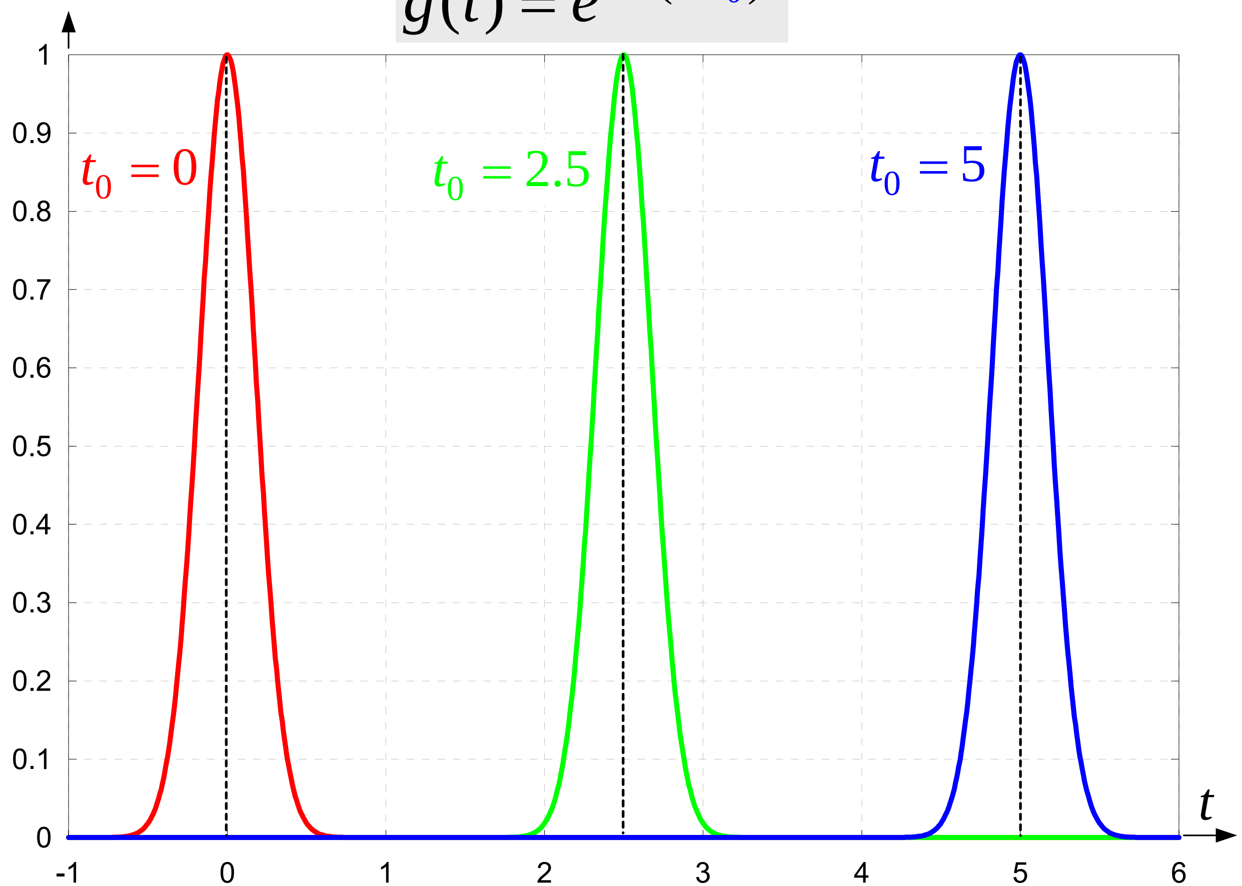
Gaussian pulse

$$g(t) = e^{-\alpha(t-t_0)^2}$$

9. Some excitation functions (sources) – cont.

Gaussian pulse

$$g(t) = e^{-\alpha(t-t_0)^2}$$

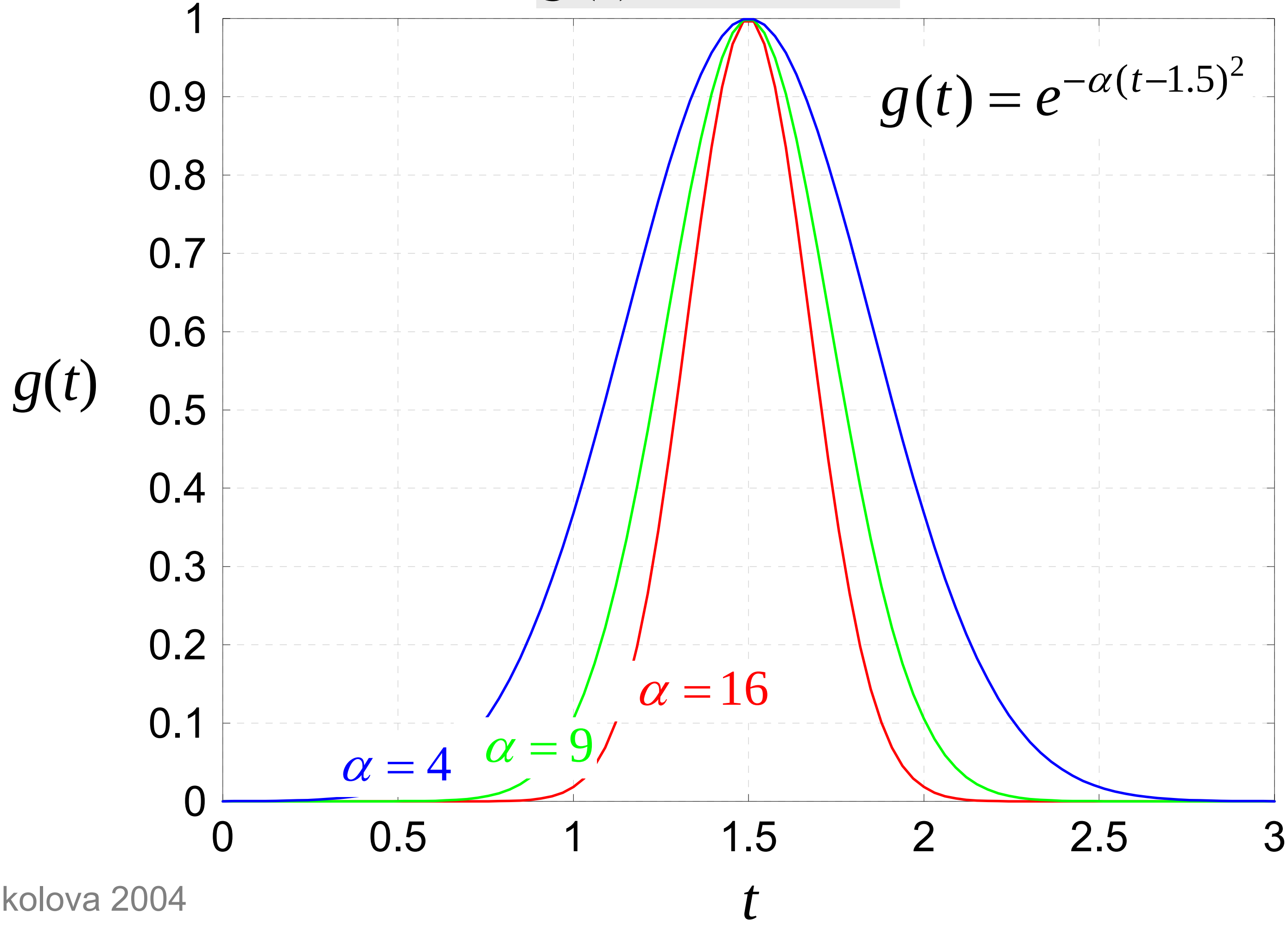


$$\boxed{f(t) = \mathrm{e}^{-16(t-t_0)^2}}$$

9. Some excitation functions (sources) – cont.

Gaussian pulse

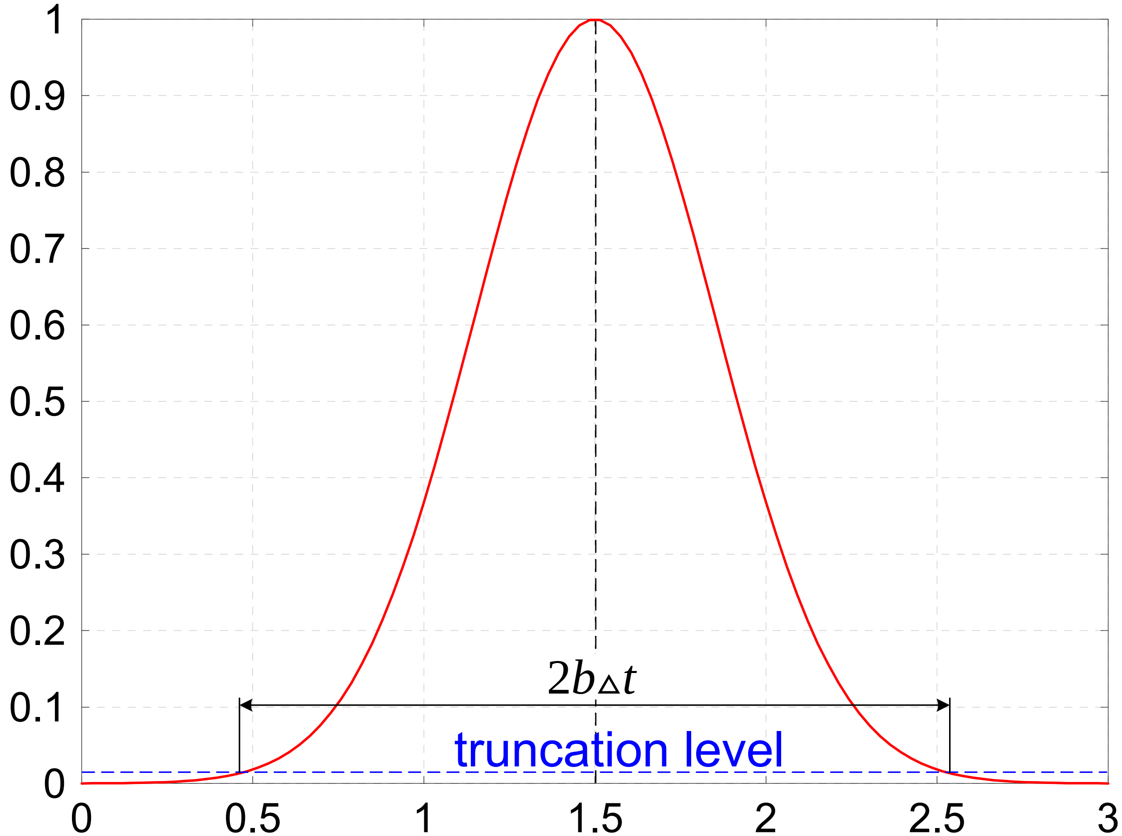
$$g(t) = e^{-\alpha (t - t_0)^2}$$



9. Some excitation functions (sources) – cont.

$$g^n = e^{-\alpha \Delta t^2 (n-n_0)^2}$$

The discrete Gaussian excitation is controlled by two numerical constants: α and b, where b denotes the half pulse width. Recommended b : $b > 30$.



9. Some excitation functions (sources) – cont.

At the truncation level: $|n - n_0| = b$

Note: $n_0 \geq b$

Set truncation level at $e^{-\eta^2}$

$$e^{-\alpha \triangle t^2 b^2} = e^{-\eta^2} \Rightarrow \boxed{\alpha \triangle t^2 = A = (\eta / b)^2} \quad \Rightarrow \quad g^n = e^{-A(n-n_0)^2}$$

The truncation level should be comparable with the precision of numbers. For single precision (6 significant digits), recommended value is $\eta = 4$. This corresponds to truncation at exp(-16). For double precision (12 significant digits), $\eta = 5$.

The pulse width in terms of b is determined according to the desired width of the excitation frequency spectrum.

9. Some excitation functions (sources) – cont.

The Fourier transform of the Gaussian pulse is a Gaussian function of frequency:

$$\tilde{g}(f) \simeq e^{-\pi^2 f^2/\alpha}$$

We require that the spectral value at the highest frequency of interest is at 0.3 of the maximum:

$$e^{-\pi^2 f_{\max}^2/\alpha} = 0.3$$

$$-\frac{\pi^2 f_{\max}^2}{\alpha} = \ln 0.3 \Rightarrow A = \frac{\pi^2 f_{\max}^2 \Delta t^2}{\ln(10/3)} = \left(\frac{\eta}{b}\right)^2$$

$$\Rightarrow \quad \boxed{b = \frac{\eta\sqrt{\ln(10/3)}}{\pi f_{\max} \Delta t}}$$

If we set $f_{\max} \Delta t = \Delta t / T_{\min} = 1/32$ then $b = \dfrac{\eta 32\sqrt{\ln(10/3)}}{\pi}$

9. Some excitation functions (sources) – cont.

Typically, the spatial step Δh is first determined according to the finest detail of the structure. Then, Δt is computed such that

$$\alpha = \frac{c\Delta t}{\Delta h} \le 1/\sqrt{D}$$

where D denotes the dimensionality of the problem (D=1,2,3). Often, Δt computed this way is much smaller than $T_{min}/32$.

Then, b is determined as

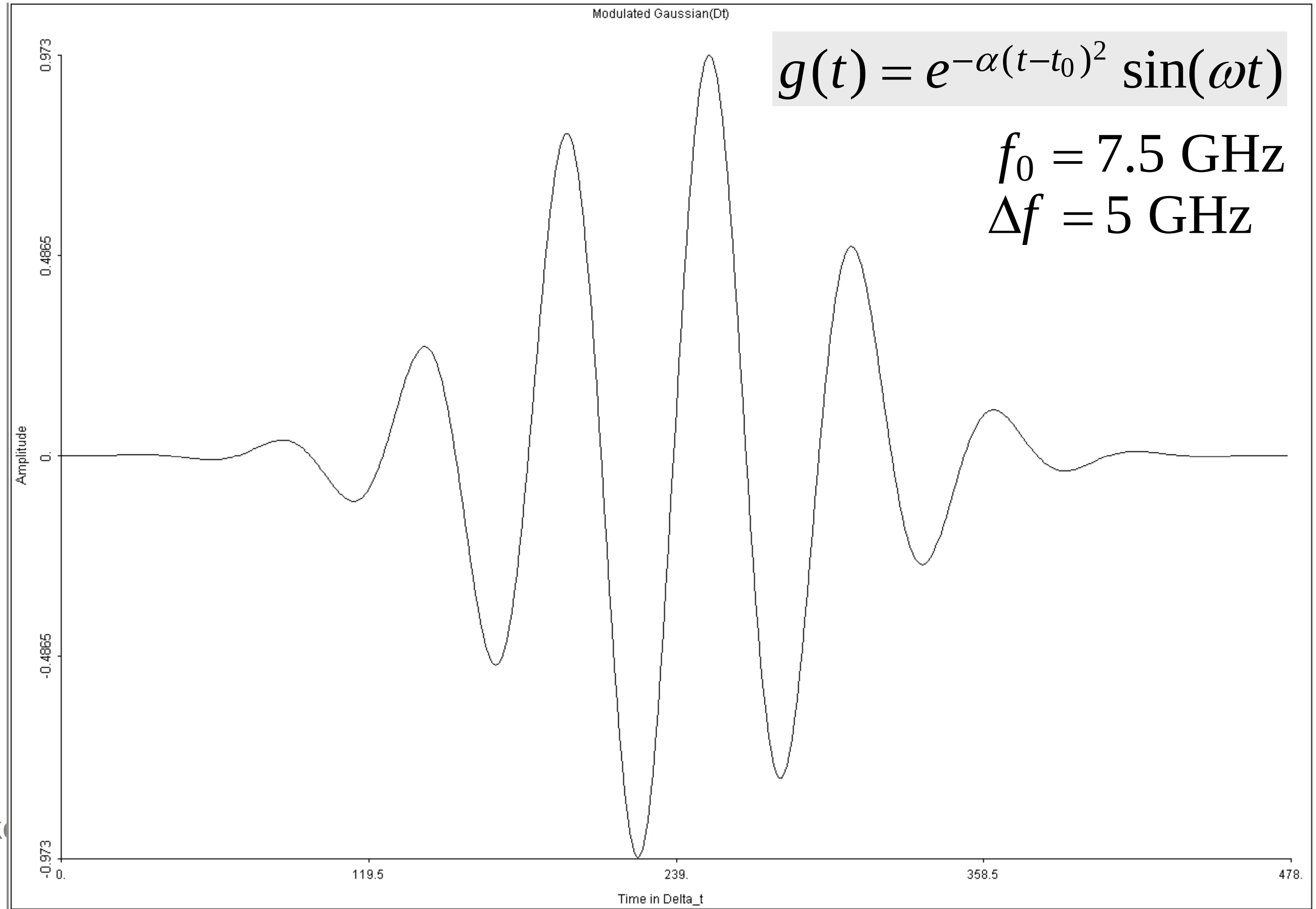
$$b = \frac{\eta\sqrt{\ln(10/3)}}{\pi\alpha}\frac{\lambda_{\min}}{\Delta h}$$

Note that for good accuracy, there should be at least 32 Δh in one λ_{min}. Often, however, the finest detail in a microwave structure is much smaller than $\lambda_{min}/2$.

9. Some excitation functions (sources) – cont.

Band-limited excitations

(a) sine wave modulated with a Gaussian pulse

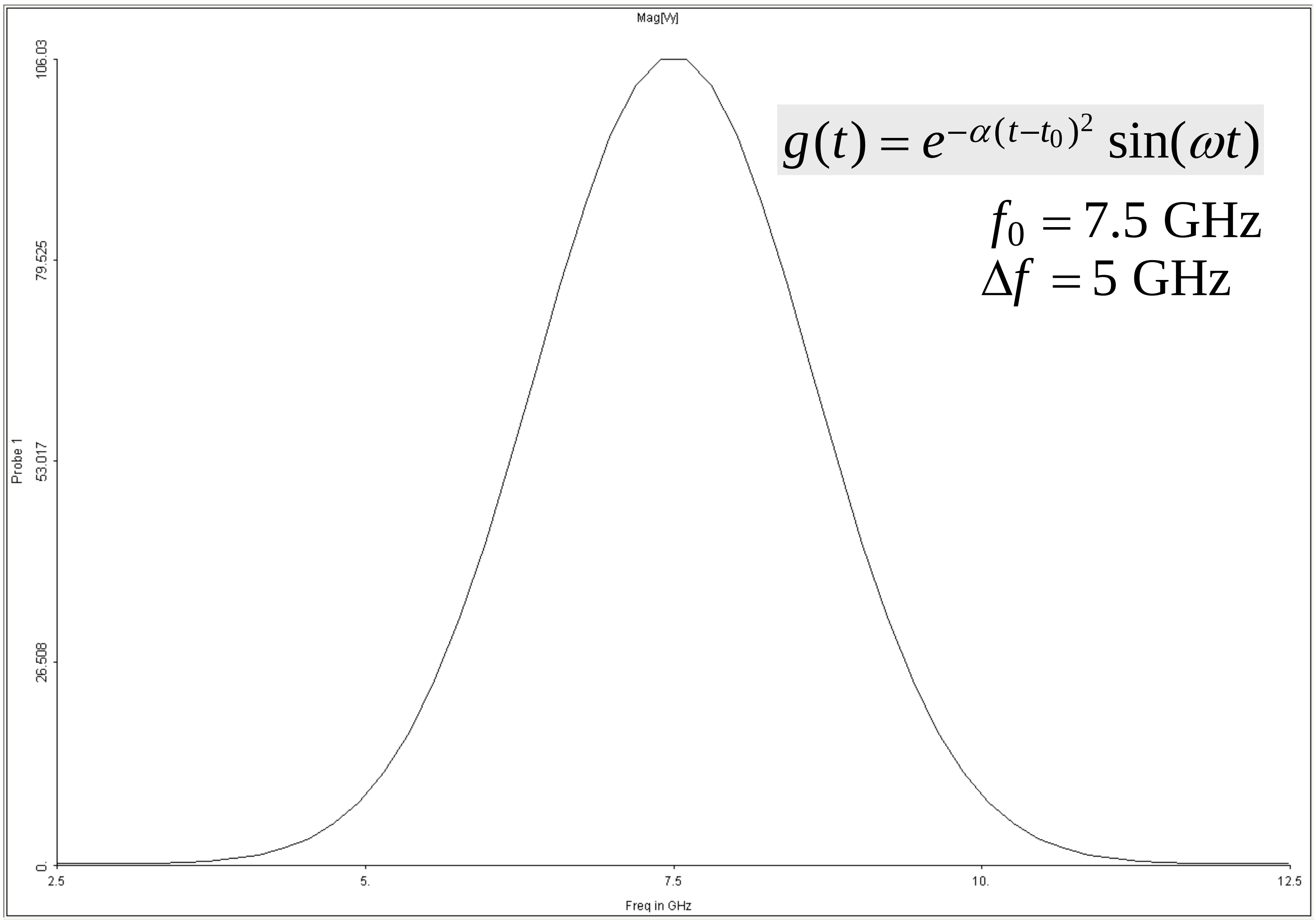


$$g(t) = e^{-\alpha(t-t_0)^2}\sin(\omega t)$$

$$f_0 = 7.5\ \text{GHz}$$
$$\Delta f = 5\ \text{GHz}$$

9. Some excitation functions (sources) – cont.

Sine wave modulated with a Gaussian pulse, cont.



$$g(t) = e^{-\alpha(t-t_0)^2} \sin(\omega t)$$

$$f_0 = 7.5 \text{ GHz}$$
$$\Delta f = 5 \text{ GHz}$$

9. Some excitation functions (sources) – cont.

(b) sine wave modulated by a Blackman-Harris window

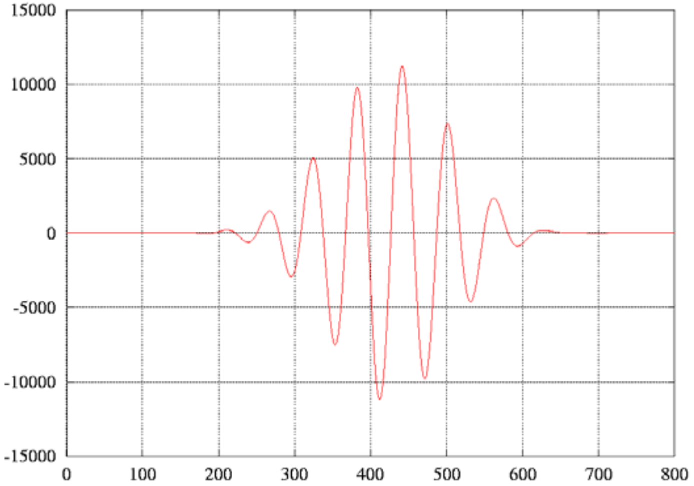
$$g(t) = B(t)\sin(\omega t),$$

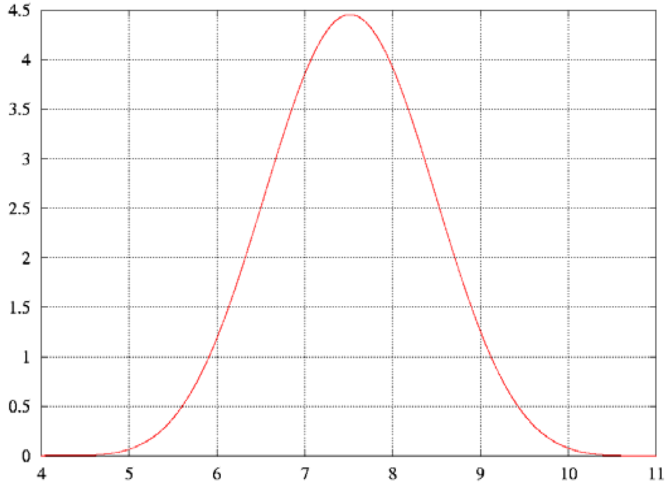
$$B(t) = a_0 - a_1 \cdot \cos\left(\omega_w t\right) + a_2 \cdot \cos\left(2\omega_w t\right) - a_3 \cdot \cos\left(3\omega_w t\right),$$

$$\omega_w = \omega / N \quad (N = 7\ldots10)$$

$$a_0 = 0.35875 \quad a_1 = 0.48829 \quad a_2 = 0.14128 \quad a_3 = 0.01168$$

N controls the bandwidth of the spectrum.





10. Time-domain Maxwell equations for a dispersion-free medium

The FDTD algorithm is based on the Maxwell curl equations

$$-\nabla\times\mathbf{E}(\mathbf{r},t)=\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}+\mathbf{J}_m(\mathbf{r},t)+\mathbf{J}_m^i(\mathbf{r},t)$$

$$\nabla\times\mathbf{H}(\mathbf{r},t)=\frac{\partial\mathbf{D}(\mathbf{r},t)}{\partial t}+\mathbf{J}_e(\mathbf{r},t)+\mathbf{J}_e^i(\mathbf{r},t)$$

<u>constitutive relations</u>

$$\mathbf{D}=\boldsymbol{F}_D\left\{\mathbf{E},\mathbf{H}\right\}$$

$$\mathbf{B}=\boldsymbol{F}_B\left\{\mathbf{E},\mathbf{H}\right\}$$

$$\mathbf{J}_e=\boldsymbol{F}_J^e\left\{\mathbf{E},\mathbf{H}\right\}$$

$$\mathbf{J}_m=\boldsymbol{F}_J^m\left\{\mathbf{E},\mathbf{H}\right\}$$

<u>in dispersion-free isotropic medium</u>

$$\mathbf{D}(\mathbf{r},t)=\varepsilon\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{B}(\mathbf{r},t)=\mu\mathbf{H}(\mathbf{r},t)$$

$$\mathbf{J}_e(\mathbf{r},t)=\sigma_e\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{J}_m(\mathbf{r},t)=\sigma_m\mathbf{H}(\mathbf{r},t)$$

10. Time-domain Maxwell equations for a dispersion-free medium – cont.

$$
\begin{aligned}
-\nabla \times \mathbf{E}(\mathbf{r},t) &= \mu \frac{\partial \mathbf{H}(\mathbf{r},t)}{\partial t} + \sigma_m \mathbf{H}(\mathbf{r},t) + \mathbf{J}_m^i(\mathbf{r},t) \\
\nabla \times \mathbf{H}(\mathbf{r},t) &= \varepsilon \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} + \sigma_e \mathbf{E}(\mathbf{r},t) + \mathbf{J}_e^i(\mathbf{r},t)
\end{aligned}
$$

In rectangular coordinates, the above is written as

$$
\begin{aligned}
\mu \frac{\partial H_x}{\partial t} + \sigma_m H_x &= -\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) - J_{mx}^i \\
\mu \frac{\partial H_y}{\partial t} + \sigma_m H_y &= -\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) - J_{my}^i \\
\mu \frac{\partial H_z}{\partial t} + \sigma_m H_z &= -\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) - J_{mz}^i
\end{aligned}
$$

10. Time-domain Maxwell equations for a dispersion-free medium – cont.

$$\varepsilon \frac{\partial E_x}{\partial t} + \sigma_e E_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - J_{ex}^i$$
$$\varepsilon \frac{\partial E_y}{\partial t} + \sigma_e E_y = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - J_{ey}^i$$
$$\varepsilon \frac{\partial E_z}{\partial t} + \sigma_e E_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - J_{ez}^i$$

Reduction to 2-D problems

Consider a field and its sources, which *do not depend on one of the spatial coordinates*, e.g., the x coordinate. Maxwell's equation then reduce to two decoupled systems of equations, each of which involves only three field components.

10. Time-domain Maxwell equations for a dispersion-free medium – cont.

$$\boxed{\frac{\partial}{\partial x} = 0}$$

example: TE_{0m} modes in a rectangular waveguide

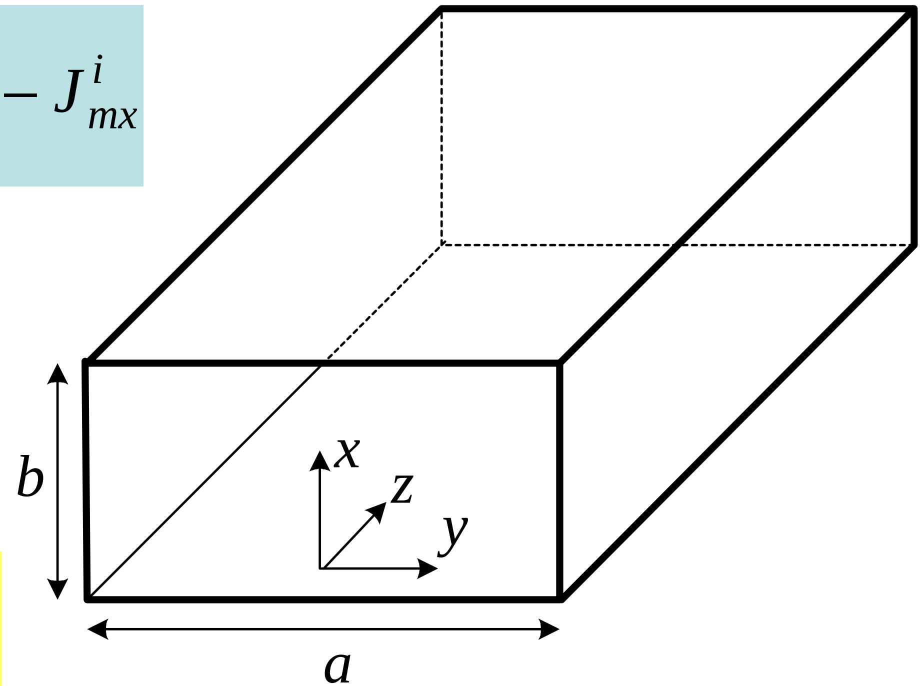
$$\mu \frac{\partial H_x}{\partial t} + \sigma_m H_x = -\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) - J^i_{mx}$$

$$\mu \frac{\partial H_y}{\partial t} + \sigma_m H_y = -\frac{\partial E_x}{\partial z} - J^i_{my}$$

$$\mu \frac{\partial H_z}{\partial t} + \sigma_m H_z = \frac{\partial E_x}{\partial y} - J^i_{mz}$$

$$\varepsilon \frac{\partial E_x}{\partial t} + \sigma_e E_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - J^i_{ex}$$

$$\varepsilon \frac{\partial E_y}{\partial t} + \sigma_e E_y = \frac{\partial H_x}{\partial z} - J^i_{ey}$$



$$\varepsilon \frac{\partial E_z}{\partial t} + \sigma_e E_z = -\frac{\partial H_x}{\partial y} - J^i_{ez}$$

10. Time-domain Maxwell equations for a dispersion-free medium – cont.

TE$_x$ modes

$$\mu\frac{\partial H_x}{\partial t}+\sigma_m H_x=-\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)-J^i_{mx}$$

$$\varepsilon\frac{\partial E_y}{\partial t}+\sigma_e E_y=\frac{\partial H_x}{\partial z}-J^i_{ey}$$

$$\varepsilon\frac{\partial E_z}{\partial t}+\sigma_e E_z=-\frac{\partial H_x}{\partial y}-J^i_{ez}$$

TM$_x$ modes

$$\varepsilon\frac{\partial E_x}{\partial t}+\sigma_e E_x=\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}-J^i_{ex}$$

$$\mu\frac{\partial H_y}{\partial t}+\sigma_m H_y=-\frac{\partial E_x}{\partial z}-J^i_{my}$$

$$\mu\frac{\partial H_z}{\partial t}+\sigma_m H_z=\frac{\partial E_x}{\partial y}-J^i_{mz}$$

10. Time-domain Maxwell equations for a dispersion-free medium – cont.

Important note: The field decomposition into TE and TM modes in 2-D problems is different from the 3-D TE and TM decomposition you know from waveguide theory!

Consider the TE_z and TM_z modes in a waveguide and the TE_z and TM_z modes in a 2-D problem.

(a) In 2-D problems – the field is independent of *z*.

(b) In a waveguide – the field depends on *z*.

(c) In 2-D problems – the 2-D field has only 3 nonzero components (TE_z: *Ex*,*Ey*,*Hz*; TM_z: *Hx*,*Hy*,*Ez*).

(d) In a waveguide – the 3-D modal field has 5 nonzero components (TEz: *Ez*=0; TMz: *Hz*=0)

10. Time-domain Maxwell equations for a dispersion-free medium – cont.

The 2-D TE_z mode

$$\mu \frac{\partial H_z}{\partial t} + \sigma_m H_z = -\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

$$\varepsilon \frac{\partial E_x}{\partial t} + \sigma_e E_x = \frac{\partial H_z}{\partial y} - J_{ex}^{i}$$

$$\varepsilon \frac{\partial E_y}{\partial t} + \sigma_e E_y = -\frac{\partial H_z}{\partial x} - J_{ey}^{i}$$

The 3-D TE_z mode

$$\boxed{\mathbf{F} = \hat{\mathbf{z}} F(x, y, z, t)}$$

$$\mathbf{E} = -\nabla \times \mathbf{F}$$

$$\mu \frac{\partial \mathbf{H}}{\partial t} + \sigma_m \mathbf{H} = \nabla \times \nabla \times \mathbf{F}$$

$$E_z = 0$$

$$E_x = -\frac{\partial F}{\partial y}; \quad E_y = \frac{\partial F}{\partial x}$$

$$\mu \frac{\partial H_z}{\partial t} + \sigma_m H_z = \frac{\partial^2 F}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 F}{\partial t^2} = -\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right)$$

$$\mu \frac{\partial H_x}{\partial t} + \sigma_m H_x = \frac{\partial^2 F}{\partial x \partial z}$$

$$\mu \frac{\partial H_y}{\partial t} + \sigma_m H_y = \frac{\partial^2 F}{\partial y \partial z}$$

10. Time-domain Maxwell equations for a dispersion-free medium – cont.

The 2-D TM$_z$ mode

$$\varepsilon \frac{\partial E_z}{\partial t} + \sigma_e E_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - J_{ez}^i$$

$$\mu \frac{\partial H_x}{\partial t} + \sigma_m H_x = -\frac{\partial E_z}{\partial y}$$

$$\mu \frac{\partial H_y}{\partial t} + \sigma_m H_y = \frac{\partial E_z}{\partial x}$$

The 3-D TM$_z$ mode

$$\boxed{\mathbf{A} = \hat{\mathbf{z}} A(x, y, z, t)}$$

$$\mathbf{H} = \nabla \times \mathbf{A}$$

$$\varepsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma_e \mathbf{E} = \nabla \times \nabla \times \mathbf{A}$$

$$H_z = 0$$

$$H_x = \frac{\partial A}{\partial y}; \quad H_y = -\frac{\partial A}{\partial x}$$

$$\varepsilon \frac{\partial E_x}{\partial t} + \sigma_e E_x = \frac{\partial^2 A}{\partial x \partial z}$$

$$\varepsilon \frac{\partial E_y}{\partial t} + \sigma_e E_y = \frac{\partial^2 A}{\partial y \partial z}$$

$$\varepsilon \frac{\partial E_z}{\partial t} + \sigma_e E_z = \frac{\partial^2 A}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right)$$

10. Time-domain Maxwell equations for a dispersion-free medium – cont.

Take as an example the TE_{z0m} waveguide modes

$$E_z = 0$$

$$E_x = -\frac{\partial F}{\partial y}; \quad E_y = \frac{\partial F}{\partial x} = 0$$

$$\mu \frac{\partial H_y}{\partial t} + \sigma_m H_y = \frac{\partial^2 F}{\partial y \partial z}$$

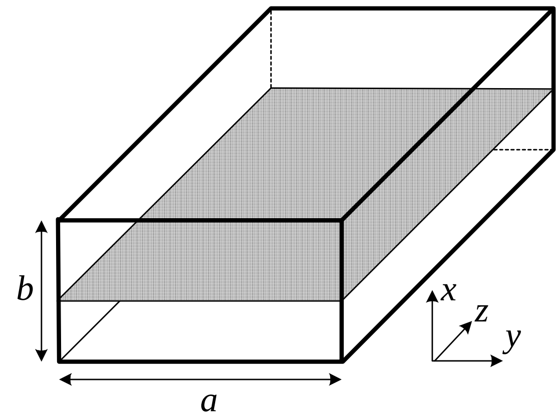
$$\mu \frac{\partial H_x}{\partial t} + \sigma_m H_x = \frac{\partial^2 F}{\partial x \partial z} = 0$$

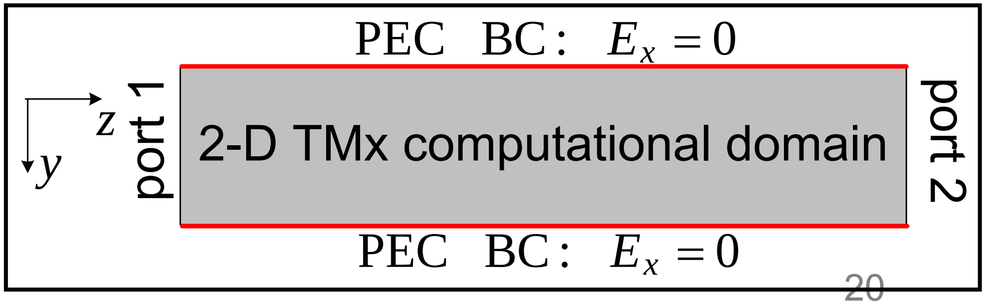
$$\mu \frac{\partial H_z}{\partial t} + \sigma_m H_z = \frac{\partial^2 F}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}$$

$$= -\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)$$

$\underline{E_x, H_y, H_z}$

This is a 2-D TM_x mode!





10. Time-domain Maxwell equations for a dispersion-free medium – cont.

There are no TM$_{z0m}$ waveguide modes! The boundary conditions

$$\frac{\partial A_z}{\partial x} = 0$$ (no dependence on *x*), and

$$A_z(x=0)=0,\ A_z(x=b)=0$$

make only the trivial solution possible:

$$A_z(x,y,z,t)=0$$

Thus, the field is zero, too.