Numerical Techniques in Electromagnetics

THE FINITE-DIFFERENCE TIME-DOMAIN (FDTD) METHOD – PART IV

The Perfectly Matched Layer (PML) Absorbing Boundary Conditions

1. The need for good absorbers

good performance of the absorbers is crucial for

(1) the accuracy of frequency-domain responses

(2) reducing the size of the computational domain

(3) the analysis of low-RCS targets, low-reflection coatings, matched loads, etc.

numerical errors below -40 dB (1/100) always desirable, sometimes -80 dB

Mur and Liao absorbers provide effective reflection coefficients of about 0.5 to 5 %: possible errors above -40 dB

Berenger publishes his first work on PML in 1994 reporting reflections of about 3000 times less than Mur’s 2nd order ABC!

2. Theory of plane wave diffraction: review

We know that for reflection-free propagation through the interface between two mediums, their intrinsic impedances must be matched. The intrinsic impedance of a fictitious medium which has both electric and magnetic conductivity is

$$\eta_l = \sqrt{\frac{\tilde{\mu}}{\tilde{\varepsilon}}} = \sqrt{\frac{\mu' - j\mu''}{\varepsilon' - j\varepsilon''}} \qquad \text{where} \qquad \varepsilon'' = \frac{\sigma_e}{\omega}, \ \mu'' = \frac{\sigma_m}{\omega}$$

provided there is no AC loss. Let the lossy region be region 2 onto which plane waves are incident from region 1. Region 1 is loss-free and with parameters ε, μ. Its intrinsic impedance is then $\eta = \sqrt{\mu / \varepsilon}$. Let $\varepsilon' = \varepsilon$, $\mu' = \mu$. The propagation constants are

$$\gamma = j\omega\sqrt{\mu\varepsilon} \qquad \text{and} \qquad \gamma_l = j\omega\sqrt{\varepsilon\mu}\sqrt{\left(1 - j\frac{\sigma_e}{\omega\varepsilon}\right)\left(1 - j\frac{\sigma_m}{\omega\mu}\right)}$$

2. Theory of plane wave diffraction, cont.

If the condition

$$\frac{\sigma_e}{\varepsilon} = \frac{\sigma_m}{\mu}$$ ⚠

impedance
matching condition

is observed in the lossy medium, then $\eta_l = \eta$, and ***a plane wave normally incident upon the interface is not reflected back***! Moreover, the velocity of propagation is the same as in region 1:

$$\gamma_l = j\underbrace{\omega\sqrt{\mu\varepsilon}}_{\beta} + \underbrace{\eta\sigma_e}_{\alpha}$$

and the medium is dispersion-free despite its losses.

2. Theory of plane wave diffraction, cont.

At oblique incidence, it is not enough to ensure that the impedance matching condition is observed (even if this condition is observed for any angle of incidence). For example, recollect that the reflection coefficients for perpendicular and parallel polarization of the wave are

$$\Gamma_{\perp} = \frac{E^{r}_{\perp 0}}{E^{i}_{\perp 0}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \qquad \Gamma_{\parallel} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

The angles of incidence and transmission Θ_i and Θ_t are related through the phase matching condition:

$$\gamma_1 \sin\theta_i = \gamma_2 \sin\theta_t$$

which ensures the *continuity of the tangential to the interface field components*. When $\eta_1 = \eta_2$, reflection is zero only if the angles of incidence and transmission are the same! We next see how all these conditions are observed in the PML medium.

3. Berenger's Perfectly Matched Medium: TE Case

Maxwell's equations for the TE$_z$ case (source-free):

$$\mu\frac{\partial H_z}{\partial t}+\sigma_m H_z=-\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)$$

$$\varepsilon\frac{\partial E_x}{\partial t}+\sigma_e E_x=\frac{\partial H_z}{\partial y}$$

$$\varepsilon\frac{\partial E_y}{\partial t}+\sigma_e E_y=-\frac{\partial H_z}{\partial x}$$

Berenger splits the H_z field component: $H_z=H_{zx}+H_{zy}$

so that (look at the 1st equation), the x-derivative of the electric field generates H_{zx}, and the y-derivative of the E-field generates H_{zy}.

He also introduces different specific conductivities to accompany the split terms.

3. Berenger's Perfectly Matched Medium: TE Case, cont.

$$\mu \frac{\partial H_{zx}}{\partial t} + \sigma_{mx} H_{zx} = -\frac{\partial E_y}{\partial x}$$

$$\mu \frac{\partial H_{zy}}{\partial t} + \sigma_{my} H_{zy} = \frac{\partial E_x}{\partial y}$$

$$\varepsilon \frac{\partial E_x}{\partial t} + \sigma_{ey} E_x = \frac{\partial (H_{zx} + H_{zy})}{\partial y}$$

$$\varepsilon \frac{\partial E_y}{\partial t} + \sigma_{ex} E_y = -\frac{\partial (H_{zx} + H_{zy})}{\partial x}$$

We next study the plane wave propagation in Berenger's medium.

4. Plane Waves in Berenger’s Medium: TE Case

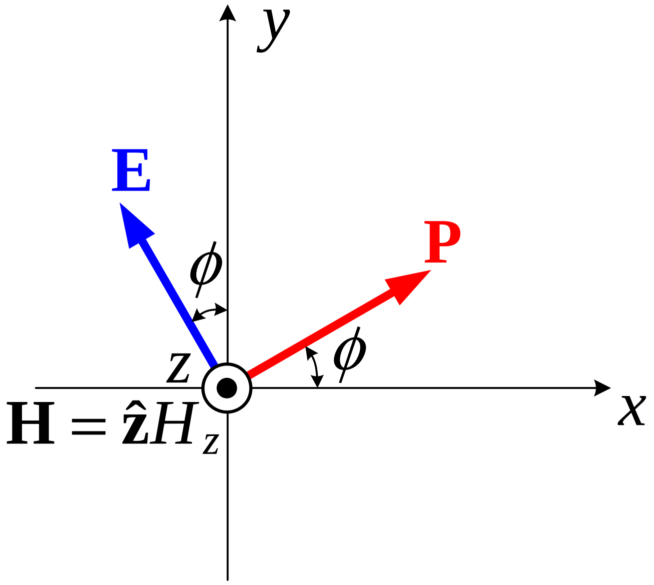
Let a time-harmonic plane TE$_z$ wave propagate as shown in the figure at an angle *Φ* with respect to the *x*-axis. The *E*-field vector then forms an angle (-*Φ*) with respect to the *y*-axis.

$$E_x = -E_0 \sin\phi \cdot e^{j\omega(t-\alpha x-\beta y)}$$

$$E_y = E_0 \cos\phi \cdot e^{j\omega(t-\alpha x-\beta y)}$$

$$H_{zx} = H_{zx0} e^{j\omega(t-\alpha x-\beta y)}$$

$$H_{zy} = H_{zy0} e^{j\omega(t-\alpha x-\beta y)}$$



The constants *α* and *β* are complex. They describe the wave behavior in space. We find them by substituting the above field components in Berenger’s TE$_z$ equations.

4. Plane Waves in Berenger's Medium: TE Case, cont.

$$\varepsilon \frac{\partial E_x}{\partial t} + \sigma_{ey} E_x = \frac{\partial (H_{zx} + H_{zy})}{\partial y} \Rightarrow \varepsilon E_0 \sin\phi - j\frac{\sigma_{ey}}{\omega} E_0 \sin\phi = \beta (H_{zx0} + H_{zy0})$$

$$\varepsilon \frac{\partial E_y}{\partial t} + \sigma_{ex} E_y = -\frac{\partial (H_{zx} + H_{zy})}{\partial x} \quad \varepsilon E_0 \cos\phi - j\frac{\sigma_{ex}}{\omega} E_0 \cos\phi = \alpha (H_{zx0} + H_{zy0})$$

$$\mu \frac{\partial H_{zx}}{\partial t} + \sigma_{mx} H_{zx} = -\frac{\partial E_y}{\partial x} \Rightarrow \mu H_{zx0} - j\frac{\sigma_{mx}}{\omega} H_{zx0} = \alpha E_0 \cos\phi$$

$$\mu \frac{\partial H_{zy}}{\partial t} + \sigma_{my} H_{zy} = \frac{\partial E_x}{\partial y} \Rightarrow \mu H_{zy0} - j\frac{\sigma_{my}}{\omega} H_{zy0} = \beta E_0 \sin\phi$$

We express H_{zx0} and H_{zy0} from the last two equations and substitute them in the 1st two.

4. Plane Waves in Berenger's Medium: TE Case, cont.

We obtain two equations for the constants *α* and *β*:

$$\frac{\mu\varepsilon}{\beta}\left(1-j\frac{\sigma_{ey}}{\omega\varepsilon}\right)\sin\phi=\frac{\alpha\cos\phi}{\left(1-j\frac{\sigma_{mx}}{\omega\mu}\right)}+\frac{\beta\sin\phi}{\left(1-j\frac{\sigma_{my}}{\omega\mu}\right)}$$

$$\frac{\mu\varepsilon}{\alpha}\left(1-j\frac{\sigma_{ex}}{\omega\varepsilon}\right)\cos\phi=\frac{\alpha\cos\phi}{\left(1-j\frac{\sigma_{mx}}{\omega\mu}\right)}+\frac{\beta\sin\phi}{\left(1-j\frac{\sigma_{my}}{\omega\mu}\right)}$$

This system gives two solution sets: we choose the one with *α* and *β* being positive so that the wave is attenuating along the positive *x* and *y* axes.

4. Plane Waves in Berenger’s Medium: TE Case, cont.

$$\alpha = \frac{1}{vG}\left(1 - j\frac{\sigma_{ex}}{\omega\varepsilon}\right)\cos\phi \qquad \beta = \frac{1}{vG}\left(1 - j\frac{\sigma_{ey}}{\omega\varepsilon}\right)\sin\phi$$

⚠

where

$$v = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad G = \sqrt{w_x \cos^2\phi + w_y \sin^2\phi},$$

$$w_x = \frac{1 - j\dfrac{\sigma_{ex}}{\omega\varepsilon}}{1 - j\dfrac{\sigma_{mx}}{\omega\mu}}, \qquad w_y = \frac{1 - j\dfrac{\sigma_{ey}}{\omega\varepsilon}}{1 - j\dfrac{\sigma_{my}}{\omega\mu}}.$$

We can now return to the system in slide 9, substitute α and β, and obtain H_{zx0} and H_{zy0} .

4. Plane Waves in Berenger’s Medium: TE Case, cont.

$$H_{zx0} = E_0 \frac{w_x \cos^2 \phi}{\eta G} \qquad H_{zy0} = E_0 \frac{w_y \sin^2 \phi}{\eta G} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\Rightarrow H_{z0} = H_{zx0} + H_{zy0} = E_0 \frac{G}{\eta}$$

Thus, the intrinsic impedance of the wave in Berenger’s PML medium is

$$\eta_{PML} = \frac{E_0}{H_{z0}} = \frac{\eta}{G}$$

⚠

4. Plane Waves in Berenger's Medium: TE Case, cont.

Each of the wave components is of the form

$$\psi = \psi_0 \exp(j\omega t) \cdot \exp\left[-j\omega \cdot \frac{1}{vG}\left(1 - j\frac{\sigma_{ex}}{\omega\varepsilon}\right)\cos\phi \cdot x \right] \times$$
$$\exp\left[-j\omega \cdot \frac{1}{vG}\left(1 - j\frac{\sigma_{ey}}{\omega\varepsilon}\right)\sin\phi \cdot y \right] .$$

Simplifying the previous expression for the wave:

$$\psi = \psi_0 \exp\left[j\omega\left(t - \frac{x\cos\phi + y\sin\phi}{vG} \right)\right] \cdot \exp\left(-\frac{\eta}{G}\sigma_{ex}\cos\phi \cdot x \right) \times$$
$$\exp\left(-\frac{\eta}{G}\sigma_{ey}\sin\phi \cdot y \right)$$

5. Impedance Match at the Interface with PML

If the conditions

$$\frac{\sigma_{ex}}{\varepsilon} = \frac{\sigma_{mx}}{\mu}, \quad \frac{\sigma_{ey}}{\varepsilon} = \frac{\sigma_{my}}{\mu}$$ ⚠

are fulfilled, then

$$w_x = w_y = 1 \qquad \Rightarrow G = \sqrt{w_x \cos^2\phi + w_y \sin^2\phi} = 1$$

$$\Rightarrow \eta_{PML} = \eta$$

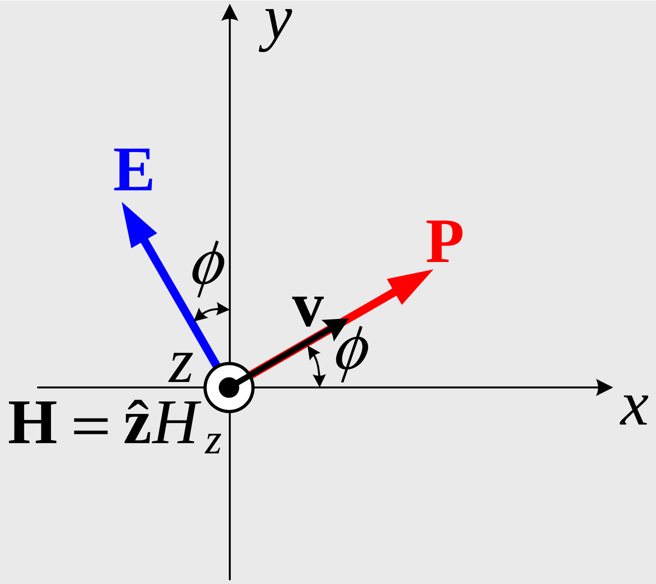
The last equation shows that the impedance of the PML medium is equal to that of the loss-free medium regardless of the angle of propagation: ***impedance match is achieved for waves of normal incidence***.

5. Impedance Match at the Interface with PML, cont.

The wave in the PML propagates as

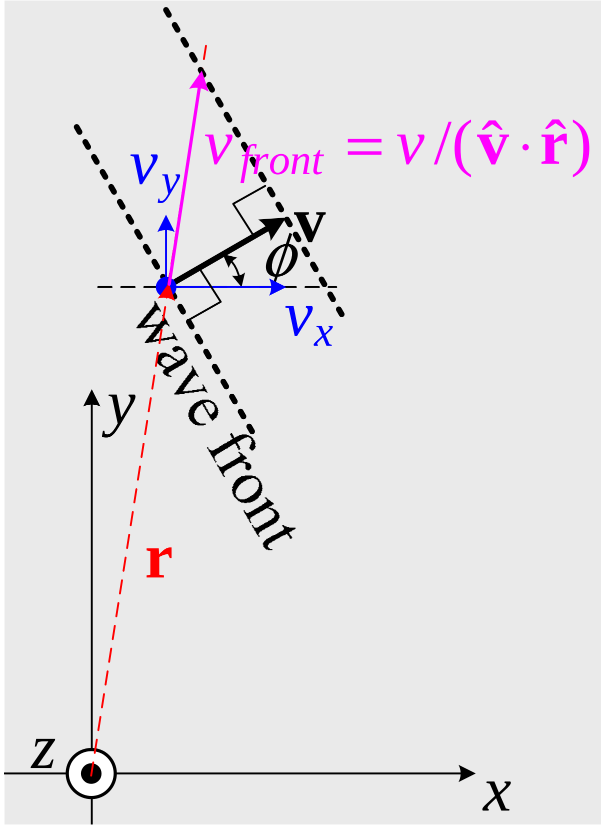
$$\psi = \psi_0 \exp\underbrace{\left[j\omega\left(t - \frac{x\cos\phi + y\sin\phi}{v}\right)\right]}_{\text{standard retardation: } j\omega(t-\mathbf{r}\cdot\mathbf{v}/v^2)} \cdot \exp\underbrace{\left[-\eta\left(\sigma_{ex}\cos\phi\cdot x + \sigma_{ey}\sin\phi\cdot y\right)\right]}_{\text{attenuation}}$$

$$\mathbf{v} = v(\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi)$$



retardation time is

$$\tau = \frac{r}{v_{front}} = \frac{r(\hat{\mathbf{v}}\cdot\hat{\mathbf{r}})}{v} = \frac{\mathbf{r}\cdot\hat{\mathbf{v}}}{v} = \frac{\mathbf{r}\cdot\mathbf{v}}{v^2}$$



5. Impedance Match at the Interface with PML, cont.

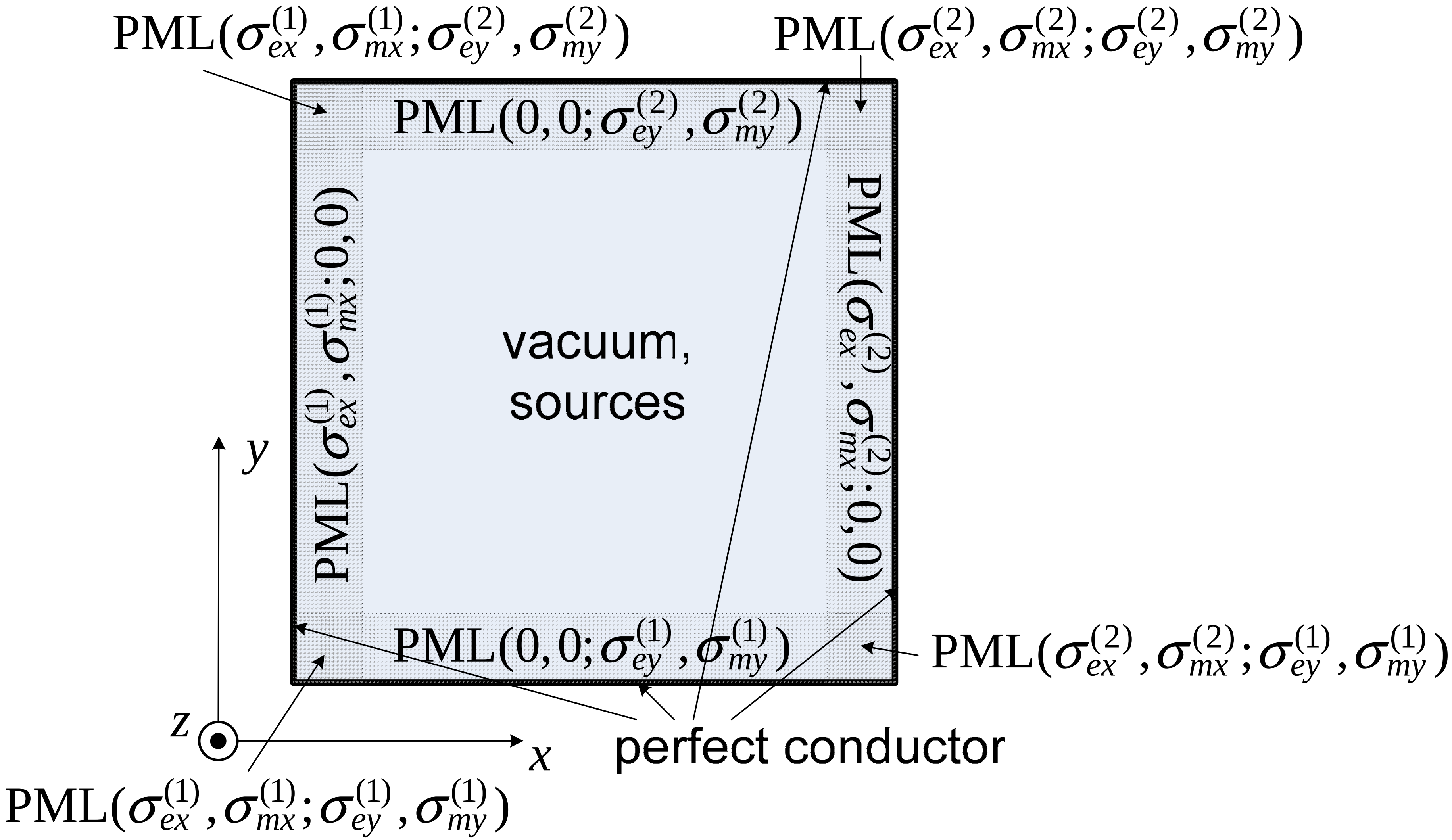
We now have to ensure that the continuity of the tangential field components is achieved by matching their phase terms along the tangential to the boundary axis. Assume that the boundary is along the *y*-axis (unit normal is ***x***). Then, the matching of the phase terms at the interface along ***y*** requires (see slide 13)

$$\exp\left(-j\omega\sqrt{\mu\varepsilon}\sin\phi\cdot y\right)=\exp\left[-j\omega\sqrt{\mu\varepsilon}\left(1-j\frac{\sigma_{ey}}{\omega\varepsilon}\right)\sin\phi\cdot y\right] \text{ at } x=0$$

This can be achieved only if $\sigma_{ey}=0$, which in accordance with the impedance-match condition means also that $\sigma_{my}=0$. There will be no attenuation along the tangential *y*-axis.

On the other hand, we require maximum attenuation along the *x*-axis. We choose appropriate functions for $\sigma_{ex}(x)$ and $\sigma_{mx}(x)$ which satisfy the impedance-match condition.

6. Berenger 2-D PML: TE$_z$ Case



$\text{PML}(\sigma_{ex}^{(1)},\sigma_{mx}^{(1)};\sigma_{ey}^{(2)},\sigma_{my}^{(2)})$

$\text{PML}(\sigma_{ex}^{(2)},\sigma_{mx}^{(2)};\sigma_{ey}^{(2)},\sigma_{my}^{(2)})$

$\text{PML}(0,0;\sigma_{ey}^{(2)},\sigma_{my}^{(2)})$

$\text{PML}(\sigma_{ex}^{(1)},\sigma_{mx}^{(1)};0,0)$

vacuum,
sources

$\text{PML}(\sigma_{ex}^{(2)},\sigma_{mx}^{(2)};0,0)$

$\text{PML}(0,0;\sigma_{ey}^{(1)},\sigma_{my}^{(1)})$

$\text{PML}(\sigma_{ex}^{(2)},\sigma_{mx}^{(2)};\sigma_{ey}^{(1)},\sigma_{my}^{(1)})$

perfect conductor

$\text{PML}(\sigma_{ex}^{(1)},\sigma_{mx}^{(1)};\sigma_{ey}^{(1)},\sigma_{my}^{(1)})$

6. Berenger 2-D PML: TE$_z$ Case, cont.

When a PML interface is orthogonal to the *x* axis (its unit normal is along *x*), the wave components must attenuate along *x*. This is accomplished by introducing σ_{ex} and σ_{mx}. To ensure the continuity of the tangential field components, σ_{ey} *and* σ_{my} must be zero.

On the contrary, for an interface of unit vector along *y*, nonzero σ_{ey} *and* σ_{my} are introduced, while σ_{ex} and σ_{mx} *are zero.*

At corner regions, all four loss parameters are nonzero.

7. Berenger 2-D PML: TM$_z$ Case

The analysis for the TE case can be repeated for a TM$_z$ wave, and it follows along the same lines. The results are completely dual. We give a summary below.

The split equations for the TM$_z$ case are

$$\varepsilon \frac{\partial E_{zx}}{\partial t} + \sigma_{ex} E_{zx} = \frac{\partial H_y}{\partial x}$$

$$\mu \frac{\partial H_x}{\partial t} + \sigma_{my} H_x = -\frac{\partial (E_{zx} + E_{zy})}{\partial y}$$

$$\varepsilon \frac{\partial E_{zy}}{\partial t} + \sigma_{ey} E_{zy} = -\frac{\partial H_x}{\partial y}$$

$$\mu \frac{\partial H_y}{\partial t} + \sigma_{mx} H_y = \frac{\partial (E_{zx} + E_{zy})}{\partial x}$$

The PML matching conditions are the same, and the 2-D PML regions are constructed as in slide 17.

8. Berenger’s 3-D PML

In 3-D, all six field components are split according to the field component derivatives generating them. The procedure of splitting is identical to the 2-D cases.

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_{ey}\right)E_{xy}=\frac{\partial}{\partial y}\left(H_{zx}+H_{zy}\right)$$

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_{ez}\right)E_{xz}=-\frac{\partial}{\partial z}\left(H_{yx}+H_{yz}\right)$$

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_{ex}\right)E_{yx}=-\frac{\partial}{\partial x}\left(H_{zx}+H_{zy}\right)$$

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_{ez}\right)E_{yz}=\frac{\partial}{\partial z}\left(H_{xy}+H_{xz}\right)$$

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_{ex}\right)E_{zx}=\frac{\partial}{\partial x}\left(H_{yx}+H_{yz}\right)$$

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_{ey}\right)E_{zy}=-\frac{\partial}{\partial y}\left(H_{xy}+H_{xz}\right)$$

$$\left(\mu\frac{\partial}{\partial t}+\sigma_{my}\right)H_{xy}=-\frac{\partial}{\partial y}\left(E_{zx}+E_{zy}\right)$$

$$\left(\mu\frac{\partial}{\partial t}+\sigma_{mz}\right)H_{xz}=\frac{\partial}{\partial z}\left(E_{yx}+E_{yz}\right)$$

$$\left(\mu\frac{\partial}{\partial t}+\sigma_{mx}\right)H_{yx}=\frac{\partial}{\partial x}\left(E_{zx}+E_{zy}\right)$$

$$\left(\mu\frac{\partial}{\partial t}+\sigma_{mz}\right)H_{yz}=-\frac{\partial}{\partial z}\left(E_{xy}+E_{xz}\right)$$

$$\left(\mu\frac{\partial}{\partial t}+\sigma_{mx}\right)H_{zx}=-\frac{\partial}{\partial x}\left(E_{yx}+E_{yz}\right)$$

$$\left(\mu\frac{\partial}{\partial t}+\sigma_{my}\right)H_{zy}=\frac{\partial}{\partial y}\left(E_{xy}+E_{xz}\right)$$

8. Berenger's 3-D PML, cont.

The matching conditions at a planar interface between the loss-free computational region and the PML require that the specific conductivities along the unit normal of the interface must be nonzero and satisfying the impedance-match condition

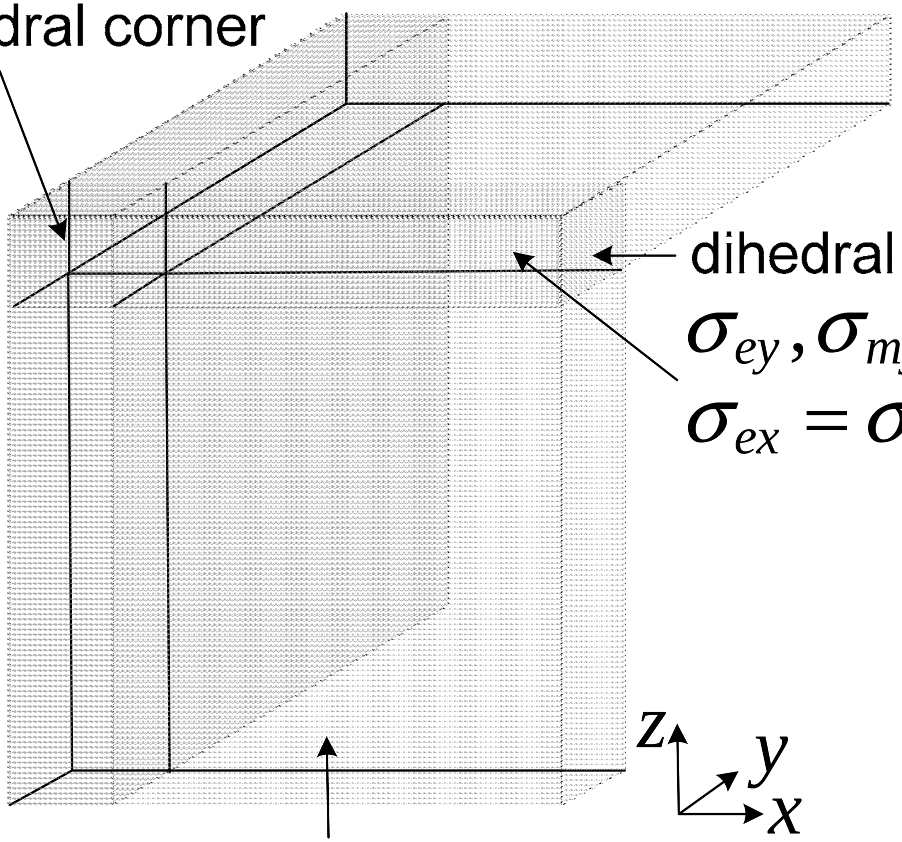
$$\frac{\sigma_{en}}{\varepsilon} = \frac{\sigma_{mn}}{\mu}$$

where *n* denotes the axis orthogonal to the planar interface. The other two pairs of conductivities (along the axes which are tangential to the interface) are set equal to zero.

In a dihedral corner where two orthogonal PMLs overlap, two pairs of conductivities are nonzero – the ones which are nonzero in the neighboring PMLs.

In a trihedral corner where three PMLs overlap, all six conductivities must be nonzero.

8. Berenger's 3-D PML, cont.



trihedral corner

$\sigma_{ey}, \sigma_{my} \neq 0$

$\sigma_{ez}, \sigma_{mz} \neq 0$

$\sigma_{ex}, \sigma_{mx} \neq 0$

dihedral corner

$\sigma_{ey}, \sigma_{my}, \sigma_{ez}, \sigma_{mz} \neq 0$

$\sigma_{ex} = \sigma_{mx} = 0$

$\sigma_{ey}, \sigma_{my} \neq 0$

$\sigma_{ex} = \sigma_{mx} = \sigma_{ez} = \sigma_{mz} = 0$

8. Berenger's 3-D PML, cont.

Discrete form of the PML equations (example for the E_{xy}, H_{xy}):

$$\hat{\mathbf{n}}=\hat{\mathbf{y}}$$

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_{ey}\right)E_{xy}=\frac{\partial}{\partial y}\left(H_{zx}+H_{zy}\right)$$

$$E^{n+1}_{xy\,i,j,k}=k_E^E\cdot E^{n}_{xy\,i,j,k}+k_H^E\cdot\left(\frac{H^{n+0.5}_{z\,i,j,k}-H^{n+0.5}_{z\,i,j-1,k}}{\triangle y}\right)$$

$$k_E^E=\frac{1-\xi_e}{1+\xi_e},\;k_H^E=\frac{\triangle t/\varepsilon}{1+\xi_e},$$

$$\xi_e=\frac{\sigma_{ey}\left(y=j\Delta y\right)\triangle t}{2\varepsilon}$$

$$\left(\mu\frac{\partial}{\partial t}+\sigma_{my}\right)H_{xy}=-\frac{\partial}{\partial y}\left(E_{zx}+E_{zy}\right)$$

$$H^{n+0.5}_{xy\,i,j,k}=k_H^H H^{n-0.5}_{xy\,i,j,k}-k_E^H\cdot\left(\frac{E^{n}_{z\,i,j+1,k}-E^{n}_{z\,i,j,k}}{\triangle y}\right)$$

$$k_H^H=\frac{1-\xi_m}{1+\xi_m},\;k_H^E=\frac{\triangle t/\mu}{1+\xi_m},$$

$$\xi_m=\frac{\sigma_{my}\left(y=(j+1/2)\Delta y\right)\triangle t}{2\mu}$$

8. Berenger's 3-D PML, cont.

Discrete form of the PML equations as first proposed by Berenger, exponential time stepping (example for the E_{xy}, H_{xy}):

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_{ey}\right)E_{xy}=\frac{\partial}{\partial y}\left(H_{zx}+H_{zy}\right)$$

$$k_E^E=e^{-\sigma_{ey}\Delta t/\varepsilon},k_H^E=\frac{1-k_E^E}{\sigma_{ey}\Delta y}$$

$$E^{n+1}_{xy_{i,j,k}}=k_E^E\cdot E^{n}_{xy_{i,j,k}}+k_H^E\cdot\left(H^{n+0.5}_{z_{i,j,k}}-H^{n+0.5}_{z_{i,j-1,k}}\right)$$

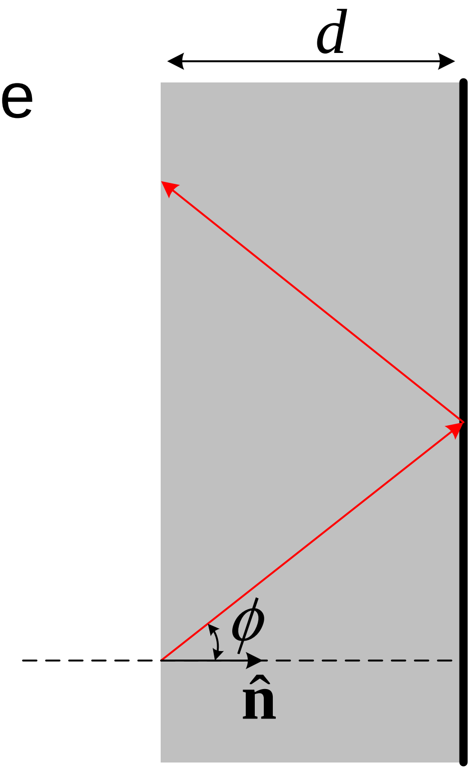
$$\left(\mu\frac{\partial}{\partial t}+\sigma_{my}\right)H_{xy}=-\frac{\partial}{\partial y}\left(E_{zx}+E_{zy}\right)$$

$$k_H^H=e^{-\sigma_{my}\Delta t/\mu},k_H^E=\frac{1-k_H^H}{\sigma_{my}\Delta y}$$

$$H^{n+0.5}_{xy_{i,j,k}}=k_H^H H^{n-0.5}_{xy_{i,j,k}}-k_E^H\cdot\left(E^{n}_{z_{i,j+1,k}}-E^{n}_{z_{i,j,k}}\right)$$

9. PML Loss Parameters

Theoretical reflection from the PML



The PML is usually backed by a PEC wall. The reflected signal undergoes reflection at the PML termination but also undergoes substantial attenuation corresponding to double the thickness of the PML *d*. In a PML layer where ***constant*** attenuation is assumed along the normal direction only (the tangential conductivities are zero), the reflection coefficient becomes

$$R(\phi) = \exp\left(-2\sigma_{en}\eta\cos\phi\cdot d\right)$$

Reminder (see slide 15): If ***n*** = ***x***, then

$$\psi(x,y) = \psi_0 \cdot e^{-\sigma_{ex}\eta\cos\phi\cdot x} \cdot e^{j\omega\left(t-\frac{x\cos\phi+y\sin\phi}{v}\right)}$$

9. PML Loss Parameters, cont.

$R(\Phi)$ is the PML reflection error. It gives the relative magnitude of the spurious reflected wave, which enters back into the computational domain. The larger d and σ_{en} are, the less the reflection. However, the angle of incidence Φ plays an important role, too. When Φ = 90 deg., R = 1! At this grazing angle of incidence, the PML is completely ineffective.

In practice, the Berenger's PML is placed sufficiently far from sources and guiding structures so that the plane-wave components of the field impinge upon the interface at angles smaller than 90 deg.

9. PML Loss Parameters, cont.

PML in Discrete Space

Theoretically, reflectionless wave transmission should take place through the PML interface, regardless of the local step discontinuity in the normal conductivities σ_{en} and σ_{mn}. In practice, however, spurious numerical reflections do arise, because of the finite spatial sampling of the field. Therefore, we can not set σ_{en} and σ_{mn} to be large constant numbers throughout the PML.

The conductivities are made functions of the PML depth: they have to be ***very small close to the PML interface*** (in order to ensure as little as possible spurious reflection), and then ***increase as quickly as possible toward the PEC termination wall*** (in order to ensure sufficient attenuation).

9. PML Loss Parameters, cont.

Assume that *x* is the position measured from the PML interface inward toward its PEC termination. Then, $\sigma_{ex}(x)$, and

$$R(\phi) = \exp\left(-2\eta \cos\phi \cdot \int_0^d \sigma_{en}(x) dx \right)$$

There are various profiles for the conductivity.

(a) Polynomial grading

$$\sigma_{ex} = \left(\frac{x}{d} \right)^m \sigma_{e\max} \quad \Rightarrow \sigma_{ex}(0) = 0, \ \sigma_{ex}(d) = \sigma_{e\max}$$

The bigger *m* is, the smoother the change of σ_{ex} close to the interface. But, then, the steeper its slope is close to the PEC walls: spurious numerical reflections occur.

9. PML Loss Parameters, cont.

We then have to bring down σ_{emax}. This, however, may lead to insufficient attenuation. Alternatively, we can keep σ_{emax} large but increase the PML depth *d* to allow for acceptable slopes at all points deep in the PML. This, however, means increase of the required computational resources.

Designing an efficient PML is not an easy task!

The reflection coefficient with polynomial grading is

$$R(\phi) = \exp\left[-2\eta\sigma_{e\max} d \cos\phi / (m+1)\right]$$

Typical optimal values:

$$2 \le m \le 6$$

$$R(0) \approx 10^{-16} \;\; (d = 10\Delta x),\; 10^{-8} (d = 5\Delta x)$$

9. PML Loss Parameters, cont.

When *R*(0), *m*, and *d* are set, we can compute σ_{emax}:

$$\sigma_{e\max} = -\frac{(m+1)\ln\left[R(0)\right]}{2\eta d}$$

(a) Geometric grading

The PML loss factor is defined as

$$\sigma_{ex}(x) = \left(g^{1/\Delta x}\right)^{x} \sigma_{x0} \qquad \Rightarrow \sigma_{ex}(0) = \sigma_{x0},\ \ \sigma_{ex}(d) = \sigma_{x0} g^{d/\Delta}$$

scaling factor conductivity at interface

$$\Rightarrow R(\phi) = \exp\left[-2\eta\sigma_{x0}\Delta x\left(g^{d/\Delta x} - 1\right)\cos\phi / \ln g\right]$$

9. PML Loss Parameters, cont.

σ_{x0} must be small for less spurious reflection from the interface. The scaling $g > 1$ determines the rate of increase of the conductivity. Large g's flatten the conductivity profile near the interface and make it steeper deeper into the PML. Usually,

$$2 \le g \le 3$$

If $R(0)$, g and d are given, we can compute σ_{x0}:

$$\sigma_{x0} = -\frac{\ln\left[R(0)\right]\ln g}{2\eta\Delta x\left(g^{d/\Delta x} - 1\right)}$$

9. PML Loss Parameters, cont.

There is another implementational detail concerning the computation of the conductivity at a mesh point: it is given by the average value in the cell around the index (*L*) location:

$$\sigma_{en}(L) = \frac{1}{\Delta x} \int_{(L-0.5)\Delta x}^{(L+0.5)\Delta x} \sigma_{en}(x)dx$$

Thus, for a polynomial grading of order *m* in a PML, which is *N*-cell thick,

$$\sigma_{ex}^{(m,N)}(0) = \frac{\sigma_{e\max}}{(m+1)2^{m+1}N^{m}} = -\frac{\ln\left[R(0)\right]}{2^{m+2}\eta\Delta x N^{m+1}}$$

$$\sigma_{ex}^{(m,N)}(L>0) = \sigma_{ex}^{(m,N)}(0)\cdot\left[(2L+1)^{m+1}-(2L-1)^{m+1}\right]$$

9. PML Loss Parameters, cont.

For the geometric grading of scaling *g* in a PML of *N* cells,

$$\sigma_{ex}^{(g,N)}(0) = \sigma_{e0}\,\frac{\sqrt{g}-1}{\ln g} = \frac{(1-\sqrt{g})\ln\left[R(0)\right]}{2\eta\Delta x(g^{N}-1)}$$

$$\sigma_{ex}^{(g,N)}(L>0) = \sigma_{ex}^{(g,N)}(0)\cdot g^{L-1/2}$$

Important topics not even mentioned in this course

FDTD numerical dispersion errors

FDTD on curvilinear grids

FDTD in dispersive and anisotropic media

FDTD in nonlinear and gain materials

Integrating lumped elements with the FDTD full-wave analysis

Excitation schemes for enhanced convergence

Near-to-Far-Field transformation for antenna radiation patterns

Modified implicit FDTD schemes – the FDTD-ADI