Numerical Techniques in Electromagnetics

THE FINITE-DIFFERENCE TIME-DOMAIN (FDTD) METHOD – PART II
9. Some excitation functions (sources)

The problem of choosing the proper space-time dependence of the excitation \( g(x,y,z,t) \) requires special attention. Here, we confine our attention to the time-dependence of the sources and their discretization.

**Sinusoidal excitation**

\[
\sin(\omega t) = \sin \left( n \frac{2\pi}{T} \Delta t \right), \quad n = 0,1,2,3,\ldots
\]

Assuming that \( \Delta t = T / 32 \) \( \Rightarrow \) \( g^n = \sin \left( \frac{\pi}{16} n \right) \), \( n = 0,1,2,\ldots \)

**Gaussian pulse**

\[
g(t) = e^{-\alpha (t-t_0)^2}
\]

Gaussian pulse

\[ g(t) = e^{-\alpha (t-t_0)^2} \]

\[ f(t) = e^{-16(t-t_0)^2} \]

Gaussian pulse

\[ g(t) = e^{-\alpha(t-t_0)^2} \]

\[ g^n = e^{-\alpha \Delta t^2 (n-n_0)^2} \]

The discrete Gaussian excitation is controlled by two numerical constants: \( \alpha \) and \( b \), where \( b \) denotes the half pulse width. Recommended \( b : b > 30 \).

At the truncation level: \( |n - n_0| = b \) \[ \text{Note: } n_0 \geq b \]

Set truncation level at \( e^{-\eta^2} \)

\[
e^{-\alpha t^2 b^2} = e^{-\eta^2} \implies \alpha t^2 = A = (\eta/b)^2 \quad \implies \quad g^n = e^{-A(n-n_0)^2}
\]

The truncation level should be comparable with the precision of numbers. For single precision (6 significant digits), recommended value is \( \eta = 4 \). This corresponds to truncation at \( \exp(-16) \). For double precision (12 significant digits), \( \eta = 5 \).

The pulse width in terms of \( b \) is determined according to the desired width of the excitation frequency spectrum.

The Fourier transform of the Gaussian pulse is a Gaussian function of frequency:

\[ \tilde{g}(f) \approx e^{-\pi^2 f^2 / \alpha} \]

We require that the spectral value at the highest frequency of interest is at 0.3 of the maximum:

\[ e^{-\pi^2 f_{\text{max}}^2 / \alpha} = 0.3 \]

\[ -\frac{\pi^2 f_{\text{max}}^2}{\alpha} = \ln 0.3 \Rightarrow A = \frac{\pi^2 f_{\text{max}}^2 \Delta t^2}{\ln(10/3)} = \left( \frac{\eta}{b} \right)^2 \]

\[ \Rightarrow b = \frac{\eta \sqrt{\ln(10/3)}}{\pi f_{\text{max}} \Delta t} \]

If we set \( f_{\text{max}} \Delta t = \Delta t / T_{\text{min}} = 1/32 \) then \( b = \frac{\eta 32 \sqrt{\ln(10/3)}}{\pi} \)

Nikolova 2004

Typically, the spatial step $\Delta h$ is first determined according to the finest detail of the structure. Then, $\Delta t$ is computed such that

$$\alpha = \frac{c\Delta t}{\Delta h} \leq 1/\sqrt{D}$$

where $D$ denotes the dimensionality of the problem ($D=1,2,3$). Often, $\Delta t$ computed this way is much smaller than $T_{\text{min}}/32$.

Then, $b$ is determined as

$$b = \eta\sqrt{\ln(10/3)} \frac{\lambda_{\text{min}}}{\pi\alpha \Delta h}$$

Note that for good accuracy, there should be at least $32 \Delta h$ in one $\lambda_{\text{min}}$. Often, however, the finest detail in a microwave structure is much smaller than $\lambda_{\text{min}}/2$. 

Nikolova 2004

**Band-limited excitations**

(a) sine wave modulated with a Gaussian pulse

\[ g(t) = e^{-\alpha(t-t_0)^2} \sin(\omega t) \]

\[ f_0 = 7.5 \text{ GHz} \]

\[ \Delta f = 5 \text{ GHz} \]
Sine wave modulated with a Gaussian pulse, cont.

\[ g(t) = e^{-\alpha(t-t_0)^2} \sin(\omega t) \]

\[ f_0 = 7.5 \text{ GHz} \]
\[ \Delta f = 5 \text{ GHz} \]

(b) sine wave modulated by a Blackman-Harris window

\[ g(t) = B(t) \sin(\omega t), \]
\[ B(t) = a_0 - a_1 \cdot \cos(\omega_w t) + a_2 \cdot \cos(2\omega_w t) - a_3 \cdot \cos(3\omega_w t), \]
\[ \omega_w = \omega / N \quad (N = 7 \ldots 10) \]

\[ a_0 = 0.35875 \quad a_1 = 0.48829 \quad a_2 = 0.14128 \quad a_3 = 0.01168 \]

\( N \) controls the bandwidth of the spectrum.
10. Time-domain Maxwell equations for a dispersion-free medium

The FDTD algorithm is based on the Maxwell curl equations

\[-\nabla \times \mathbf{E}(\mathbf{r}, t) = \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} + \mathbf{J}_m(\mathbf{r}, t) + \mathbf{J}_m^i(\mathbf{r}, t)\]

\[\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} + \mathbf{J}_e(\mathbf{r}, t) + \mathbf{J}_e^i(\mathbf{r}, t)\]

**constitutive relations**

\[
\begin{align*}
\mathbf{D} &= F_D \{ \mathbf{E}, \mathbf{H} \} \\
\mathbf{B} &= F_B \{ \mathbf{E}, \mathbf{H} \} \\
\mathbf{J}_e &= F_{je} \{ \mathbf{E}, \mathbf{H} \} \\
\mathbf{J}_m &= F_{jm} \{ \mathbf{E}, \mathbf{H} \}
\end{align*}
\]

**in dispersion-free isotropic medium**

\[
\begin{align*}
\mathbf{D}(\mathbf{r}, t) &= \varepsilon \mathbf{E}(\mathbf{r}, t) \\
\mathbf{B}(\mathbf{r}, t) &= \mu \mathbf{H}(\mathbf{r}, t) \\
\mathbf{J}_e(\mathbf{r}, t) &= \sigma_e \mathbf{E}(\mathbf{r}, t) \\
\mathbf{J}_m(\mathbf{r}, t) &= \sigma_m \mathbf{H}(\mathbf{r}, t)
\end{align*}
\]

\[-\nabla \times \mathbf{E}(\mathbf{r}, t) = \mu \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t} + \sigma_{m} \mathbf{H}(\mathbf{r}, t) + \mathbf{J}_{i}^{\text{m}}(\mathbf{r}, t)\]

\[\nabla \times \mathbf{H}(\mathbf{r}, t) = \varepsilon \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} + \sigma_{e} \mathbf{E}(\mathbf{r}, t) + \mathbf{J}_{e}^{i}(\mathbf{r}, t)\]

In rectangular coordinates, the above is written as

\[
\mu \frac{\partial H_{x}}{\partial t} + \sigma_{m} H_{x} = - \left( \frac{\partial E_{z}}{\partial y} - \frac{\partial E_{y}}{\partial z} \right) - J_{i}^{m_{x}}
\]

\[
\mu \frac{\partial H_{y}}{\partial t} + \sigma_{m} H_{y} = - \left( \frac{\partial E_{x}}{\partial z} - \frac{\partial E_{z}}{\partial x} \right) - J_{i}^{m_{y}}
\]

\[
\mu \frac{\partial H_{z}}{\partial t} + \sigma_{m} H_{z} = - \left( \frac{\partial E_{y}}{\partial x} - \frac{\partial E_{x}}{\partial y} \right) - J_{i}^{m_{z}}
\]

\[ \varepsilon \frac{\partial E_x}{\partial t} + \sigma_e E_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - J_{ex}^i \]

\[ \varepsilon \frac{\partial E_y}{\partial t} + \sigma_e E_y = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - J_{ey}^i \]

\[ \varepsilon \frac{\partial E_z}{\partial t} + \sigma_e E_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - J_{ez}^i \]

**Reduction to 2-D problems**

Consider a field and its sources, which *do not depend on one of the spatial coordinates*, e.g., the \( x \) coordinate. Maxwell’s equation then reduce to two decoupled systems of equations, each of which involves only three field components.

\[ \frac{\partial}{\partial x} = 0 \]

example: TE_{0m} modes in a rectangular waveguide

\[
\mu \frac{\partial H_x}{\partial t} + \sigma_m H_x = -\left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) - J_{mx}^i
\]

\[
\mu \frac{\partial H_y}{\partial t} + \sigma_m H_y = -\frac{\partial E_x}{\partial z} - J_{my}^i
\]

\[
\mu \frac{\partial H_z}{\partial t} + \sigma_m H_z = \frac{\partial E_x}{\partial y} - J_{mz}^i
\]

\[
\varepsilon \frac{\partial E_x}{\partial t} + \sigma_e E_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - J_{ex}^i
\]

\[
\varepsilon \frac{\partial E_y}{\partial t} + \sigma_e E_y = \frac{\partial H_x}{\partial z} - J_{ey}^i
\]

\[
\varepsilon \frac{\partial E_z}{\partial t} + \sigma_e E_z = -\frac{\partial H_x}{\partial y} - J_{ez}^i
\]

**TE\(_x\) modes**

\[
\mu \frac{\partial H_x}{\partial t} + \sigma_m H_x = - \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) - J_{mx}^i
\]

**TM\(_x\) modes**

\[
\varepsilon \frac{\partial E_x}{\partial t} + \sigma_e E_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - J_{ex}^i
\]

\[
\mu \frac{\partial H_y}{\partial t} + \sigma_m H_y = - \frac{\partial E_x}{\partial z} - J_{my}^i
\]

\[
\mu \frac{\partial H_z}{\partial t} + \sigma_m H_z = \frac{\partial E_x}{\partial y} - J_{mz}^i
\]

**Important note:** The field decomposition into TE and TM modes in 2-D problems is different from the 3-D TE and TM decomposition you know from waveguide theory!

Consider the TE\(_z\) and TM\(_z\) modes in a waveguide and the TE\(_z\) and TM\(_z\) modes in a 2-D problem.

(a) In 2-D problems – the field is independent of \(z\).

(b) In a waveguide – the field depends on \(z\).

(c) In 2-D problems – the 2-D field has only 3 nonzero components (TE\(_z\): \(E_x, E_y, H_z\); TM\(_z\): \(H_x, H_y, E_z\)).

(d) In a waveguide – the 3-D modal field has 5 nonzero components (TE\(_z\): \(E_z=0\); TM\(_z\): \(H_z=0\))

The 2-D TE$_z$ mode

\[ \mu \frac{\partial H_z}{\partial t} + \sigma_m H_z = - \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \]

\[ \varepsilon \frac{\partial E_x}{\partial t} + \sigma_e E_x = \frac{\partial H_z}{\partial y} - J_{ex} \]

\[ \varepsilon \frac{\partial E_y}{\partial t} + \sigma_e E_y = - \frac{\partial H_z}{\partial x} - J_{ey} \]

The 3-D TE$_z$ mode

\[ \mathbf{F} = \hat{z} \mathbf{F}(x, y, z, t) \]

\[ \mathbf{E} = -\nabla \times \mathbf{F} \]

\[ \mu \frac{\partial \mathbf{H}}{\partial t} + \sigma_m \mathbf{H} = \nabla \times \nabla \times \mathbf{F} \]

\[ E_z = 0 \]

\[ E_x = -\frac{\partial F}{\partial y}; \quad E_y = \frac{\partial F}{\partial x} \]

\[ \mu \frac{\partial H_x}{\partial t} + \sigma_m H_x = \frac{\partial^2 F}{\partial x^2} + \frac{1}{c^2} \frac{\partial^2 F}{\partial t^2} \]

\[ \mu \frac{\partial H_y}{\partial t} + \sigma_m H_y = \frac{\partial^2 F}{\partial y^2} \]

\[ \mu \frac{\partial H_z}{\partial t} + \sigma_m H_z = - \left( \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right) \]

The 2-D TM$_z$ mode

\[ \varepsilon \frac{\partial E_z}{\partial t} + \sigma_e E_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - J_{ez}^i \]

\[ \mu \frac{\partial H_x}{\partial t} + \sigma_m H_x = -\frac{\partial E_z}{\partial y} \]

\[ \mu \frac{\partial H_y}{\partial t} + \sigma_m H_y = \frac{\partial E_z}{\partial x} \]

The 3-D TM$_z$ mode

\[ A = \hat{z}A(x, y, z, t) \]

\[ \mathbf{H} = \nabla \times \mathbf{A} \]

\[ \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma_e \mathbf{E} = \nabla \times \nabla \times \mathbf{A} \]

\[ H_z = 0 \]

\[ H_x = \frac{\partial A}{\partial y} ; \quad H_y = -\frac{\partial A}{\partial x} \]

\[ \varepsilon \frac{\partial E_z}{\partial t} + \sigma_e E_z = \frac{\partial^2 A}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} \]

\[ = -\left( \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} \right) \]

Take as an example the TE_{z0m} waveguide modes

\[ E_z = 0 \]
\[ E_x = -\frac{\partial F}{\partial y}; \quad E_y = \frac{\partial F}{\partial x} = 0 \]

\[ \mu \frac{\partial H_y}{\partial t} + \sigma_m H_y = \frac{\partial^2 F}{\partial y \partial z} \]
\[ \mu \frac{\partial H_x}{\partial t} + \sigma_m H_x = \frac{\partial^2 F}{\partial x \partial z} = 0 \]
\[ \mu \frac{\partial H_z}{\partial t} + \sigma_m H_z = \frac{\partial^2 F}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 F}{\partial t^2} \]

\[ = -\left( \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right) \]

This is a 2-D TM_x mode!
There are no TM\(_{z0m}\) waveguide modes! The boundary conditions

\[ \frac{\partial A_z}{\partial x} = 0 \quad \text{(no dependence on } x\text{), and} \]

\[ A_z(x = 0) = 0, \; A_z(x = b) = 0 \]

make only the trivial solution possible:

\[ A_z(x, y, z, t) = 0 \]

Thus, the field is zero, too.