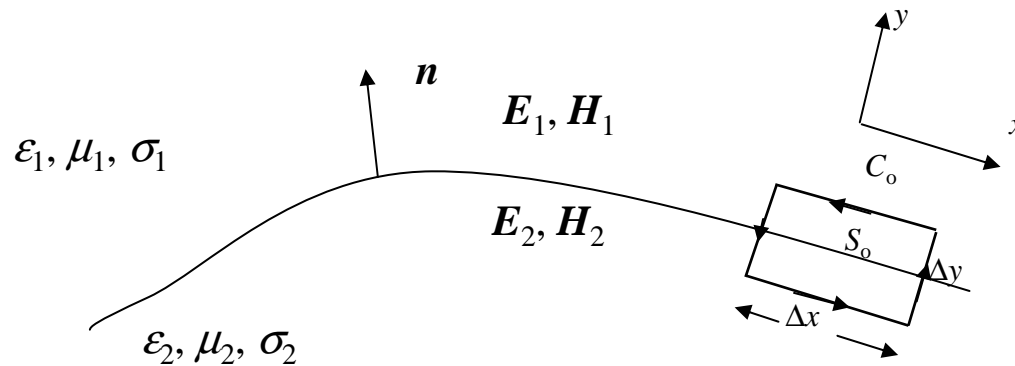


EE758
Numerical Techniques in Electromagnetics
Lecture 2

Boundary Conditions

- Maxwell's Equations are partial differential equations
- Boundary conditions are needed to obtain a unique solution
- Maxwell's differential equations do not apply on the boundaries because the fields are discontinuous
- Our target is to determine the electric and magnetic fields in a certain region of space due to excitations satisfying the problem's boundary conditions

Finite Conductivity Case



- Applying Faraday's law we get $\oint_{C_0} \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \iint_{S_0} \mathbf{B} \cdot d\mathbf{S}$
 - As $\Delta y \rightarrow 0$, the RHS vanishes and we get
- $$\mathbf{E}_1 \cdot \Delta x \mathbf{a}_x - \mathbf{E}_2 \cdot \Delta x \mathbf{a}_x = 0 \implies E_1^t = E_2^t$$

or alternatively, $\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{0}$

- It follows that the tangential component of the electric field is continuous (no magnetic current is assumed)

Finite Conductivity Case (Cont'd)

- Similarly, starting with the modified Ampere's law

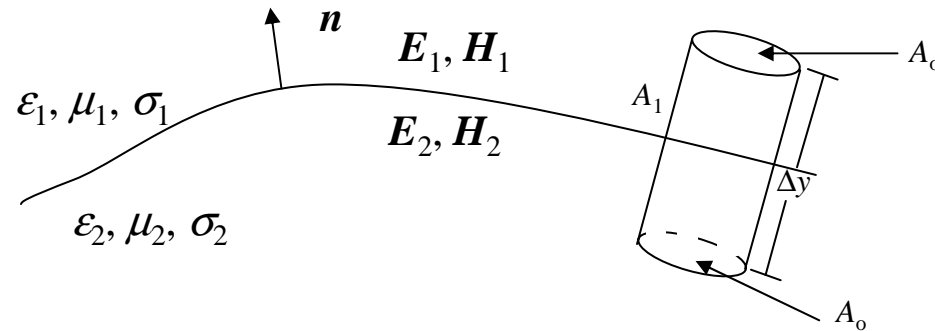
$$\oint_{C_0} \mathbf{H} \cdot d\mathbf{l} = \frac{\partial}{\partial t} \iint_{S_0} \mathbf{D} \cdot d\mathbf{S} \quad (\text{no current } \mathbf{J} \text{ at the interface), we get}$$

$$\mathbf{H}_1 \cdot \Delta x \mathbf{a}_x - \mathbf{H}_2 \cdot \Delta x \mathbf{a}_x = 0 \quad \Longrightarrow \quad H_1^t = H_2^t$$

or alternatively, $\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{0}$

- It follows that the tangential component of the magnetic field intensity is continuous if there are no boundary electric currents

Finite Conductivity Case (Cont'd)



- Assuming there are no surface charges, Gauss's law gives $\oiint_S \mathbf{D} \cdot d\mathbf{S} = Q_{ev} \implies \lim_{\Delta y \rightarrow 0} \oiint_S \mathbf{D} \cdot d\mathbf{S} = 0$
- It follows that $\mathbf{D}_1 \cdot A_o \mathbf{a}_y - \mathbf{D}_2 \cdot A_o \mathbf{a}_y = 0 \implies D_1^n = D_2^n$
or alternatively, $\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0$
- But as $D_{1,2}^n = \epsilon_{1,2} E_{1,2}^n \implies E_1^n = \frac{\epsilon_2}{\epsilon_1} E_2^n$
- Normal components of the electric field are discontinuous across the interface

Finite Conductivity Case (Cont'd)

- Similarly, by applying Gauss's law for magnetic fields we get $\lim_{\Delta y \rightarrow 0} \oiint_S \mathbf{B} \cdot d\mathbf{S} = 0$

- It follows that $\mathbf{B}_1 \cdot A_0 \mathbf{a}_y - \mathbf{B}_2 \cdot A_0 \mathbf{a}_y = 0 \implies B_1^n = B_2^n$

or alternatively, $\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$

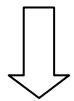
- But as $B_{1,2}^n = \mu_{1,2} H_{1,2}^n \implies H_1^n = \frac{\mu_2}{\mu_1} H_2^n$
- Normal components of the magnetic fields are discontinuous

Finite Conductivity Boundary Conditions

$\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{0}$, no interface surface magnetic currents

$\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{0}$, no interface surface electric currents

$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0$, no interface surface electric charges



$E_1^n = \frac{\epsilon_2}{\epsilon_1} E_2^n$ discontinuous normal electric field

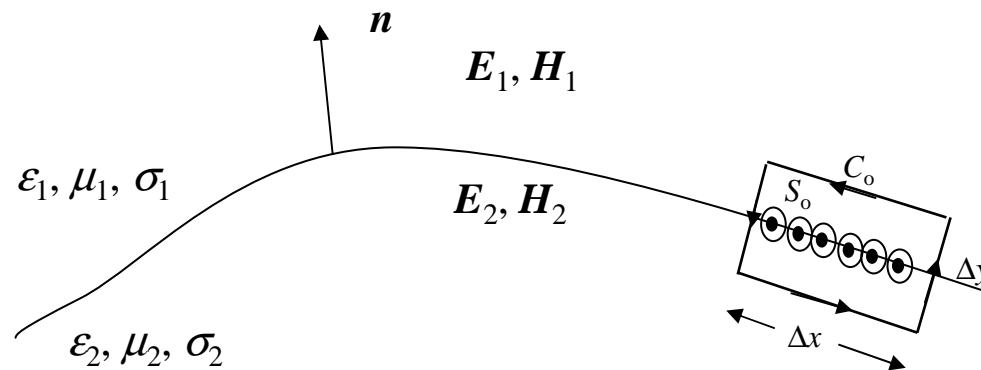
$\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$, no interface magnetic surface charges



$H_1^n = \frac{\mu_2}{\mu_1} H_2^n$ discontinuous normal magnetic field

Boundary Conditions with Sources

- Boundary conditions must be changed to take into account the existence of surface currents and surface charges



- Applying the modified Ampere's law we get

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \mathbf{J} \cdot d\mathbf{S} + \frac{\partial}{\partial t} \iint_S \mathbf{D} \cdot d\mathbf{S}$$

- Taking the limit as $\Delta y \rightarrow 0$, each integral term gives

Boundary Conditions with Sources (Cont'd)

$$\lim_{\Delta y \rightarrow 0} \oint_C \mathbf{H} \cdot d\mathbf{l} = (\mathbf{H}_2 - \mathbf{H}_1) \cdot \Delta x \mathbf{a}_x$$

$$\lim_{\Delta y \rightarrow 0} \iint_S \mathbf{D} \cdot d\mathbf{S} = 0$$

$$\lim_{\Delta y \rightarrow 0} \iint_S \mathbf{J} \cdot d\mathbf{S} = \lim_{\Delta y \rightarrow 0} \mathbf{J} \cdot \Delta x \Delta y \mathbf{a}_z = \lim_{\Delta y \rightarrow 0} (\mathbf{J} \Delta y) \cdot \Delta x \mathbf{a}_z = \lim_{\Delta y \rightarrow 0} \mathbf{J}_s \cdot \Delta x \mathbf{a}_z$$

\mathbf{J}_s is the surface current density A/m

- It follows that $(\mathbf{H}_2 - \mathbf{H}_1) \cdot \Delta x \mathbf{a}_x = \mathbf{J}_s \cdot \Delta x \mathbf{a}_z$

or alternatively $(\mathbf{H}_2 - \mathbf{H}_1) \cdot (\mathbf{a}_y \times \mathbf{a}_z) = \mathbf{J}_s \cdot \mathbf{a}_z$

$$(\mathbf{a}_y \times (\mathbf{H}_1 - \mathbf{H}_2)) \cdot \mathbf{a}_z = \mathbf{J}_s \cdot \mathbf{a}_z \quad \Longrightarrow \quad \mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

Boundary Conditions with Sources (Cont'd)

- Tangential components of the magnetic field intensity are discontinuous if surface electric current density \mathbf{J}_s (A/m) exists
- If medium 2 is a perfect conductor, we have

$$\mathbf{n} \times \mathbf{H}_1 = \mathbf{J}_s \implies H_1^t = J_s$$

- Similarly, starting with Faraday's Law

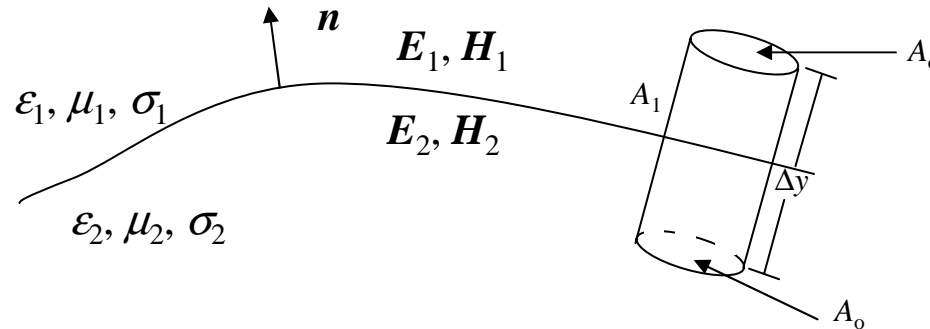
$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\iint_S \boldsymbol{\mu} \cdot d\mathbf{S} - \frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot d\mathbf{S}$$

We can reach $-\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = \boldsymbol{\mu}_s$

- For a perfect conductor we have $-\mathbf{n} \times \mathbf{E}_1 = \boldsymbol{\mu}_s$
- If no fictitious magnetic current is assumed we have

$$-\mathbf{n} \times \mathbf{E}_1 = 0 \implies E_1^t = 0$$

Boundary Conditions with Sources (Cont'd)



- Applying Gauss's law for the shown cylinder we have

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = \iiint_V q_{ev} dV \implies \lim_{\Delta y \rightarrow 0} \oiint_{A_0} \mathbf{D} \cdot d\mathbf{S} = \lim_{\Delta y \rightarrow 0} \iiint_V q_{ev} dV$$

$$(\mathbf{D}_1 - \mathbf{D}_2) \cdot A_0 \mathbf{n} = \lim_{\Delta y \rightarrow 0} (q_{ev} \Delta y) A_0 = q_{es} A_0$$

or alternatively, $D_1^n - D_2^n = q_{es}$

- Normal components of the electric flux density are discontinuous by the amount of surface charge density

Boundary Conditions with Sources (Cont'd)

- If medium 2 is a perfect conductor, we have $D_1^n = q_{es}$
- Similarly, for the magnetic flux density we may show that $n \cdot (\mathbf{B}_1 - \mathbf{B}_2) = q_{ms}$
- For perfect conductors with no magnetic charges we have $B_2^n = B_1^n = 0$

Summary of Boundary Conditions

$$\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

$$-\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = \boldsymbol{\mu}_s$$

$$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = q_{es}$$

$$\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = q_{ms}$$

Time-Harmonic Electromagnetic Fields

- If sources are sinusoidal and the medium is linear then the fields everywhere are sinusoidal as well. The field at each point is characterized by its amplitude and phase (Phasor)
- Ex: $f(x,t) = 3.0 \cos(\omega t - \beta x) = 3.0 \operatorname{Re}(\exp(j(\omega t - \beta x)))$

$$f(x,t) = \operatorname{Re}(3.0 \exp(-j\beta x) \exp(j\omega t))$$

$$f(x,t) = \operatorname{Re}(\tilde{f} \exp(j\omega t))$$

$$\tilde{f} = 3.0 \exp(-j\beta x)$$

Time-Harmonic Electromagnetic Fields (Cont'd)

- Similarly, for all field quantities we may write

$$\mathbf{E}(x, y, z, t) = \text{Re}(\tilde{\mathbf{E}}(x, y, z) \exp(j\omega t))$$

$$\mathbf{H}(x, y, z, t) = \text{Re}(\tilde{\mathbf{H}}(x, y, z) \exp(j\omega t))$$

$$\mathbf{D}(x, y, z, t) = \text{Re}(\tilde{\mathbf{D}}(x, y, z) \exp(j\omega t))$$

$$\mathbf{B}(x, y, z, t) = \text{Re}(\tilde{\mathbf{B}}(x, y, z) \exp(j\omega t))$$

$$\mathbf{J}(x, y, z, t) = \text{Re}(\tilde{\mathbf{J}}(x, y, z) \exp(j\omega t))$$

$$q(x, y, z, t) = \text{Re}(\tilde{q}(x, y, z) \exp(j\omega t))$$

Time-Harmonic Electromagnetic Fields (Cont'd)

- Maxwell's equations for the time-harmonic case are obtained by replacing each time vector by its corresponding phasor vector and replacing $\partial/\partial t$ by $j\omega$
- Maxwell's equations in the integral form are given by

$$\oiint_S \tilde{\mathbf{D}} \cdot d\mathbf{S} = \iiint_V \tilde{q}_{ev} dV = \tilde{Q}_{ev}$$

$$\oiint_S \tilde{\mathbf{B}} \cdot d\mathbf{S} = \iiint_V \tilde{q}_{mv} dV = \tilde{Q}_{mv}$$

$$\oint_C \tilde{\mathbf{E}} \cdot d\mathbf{l} = -\iint_S \tilde{\boldsymbol{\mu}} \cdot d\mathbf{S} - j\omega \iint_S \tilde{\mathbf{B}} \cdot d\mathbf{S}$$

$$\oint_C \tilde{\mathbf{H}} \cdot d\mathbf{l} = \iint_S \tilde{\mathbf{J}} \cdot d\mathbf{S} + j\omega \iint_S \tilde{\mathbf{D}} \cdot d\mathbf{S}$$

$$\oiint_S \tilde{\mathbf{J}} \cdot d\mathbf{S} = -j\omega \tilde{Q}_e$$

Time-Harmonic Electromagnetic Fields (Cont'd)

- Maxwell's equations in the differential form become

$$\nabla \cdot \tilde{\mathbf{D}} = \tilde{q}_{ev}$$

$$\nabla \cdot \tilde{\mathbf{B}} = \tilde{q}_{mv}$$

$$\nabla \cdot \tilde{\mathbf{J}} = -j\omega\tilde{q}_{ev}$$

$$(\nabla \times \tilde{\mathbf{E}}) = -\tilde{\boldsymbol{\mu}} - j\omega\tilde{\mathbf{B}}$$

$$(\nabla \times \tilde{\mathbf{H}}) = \tilde{\mathbf{J}} + j\omega\tilde{\mathbf{D}}$$

- Same boundary conditions apply

Energy and Power

- We would like to derive equations governing EM energy and power
- Starting with Maxwell's equations

$$(\nabla \times \mathbf{E}) = -\boldsymbol{\mu}_i - \frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\mu}_i - \boldsymbol{\mu}_d \quad (.H)$$

$$(\nabla \times \mathbf{H}) = \mathbf{J}_i + \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_i + \mathbf{J}_c + \mathbf{J}_d \quad (.E)$$

Subtracting we get

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) = -\mathbf{H} \cdot (\boldsymbol{\mu}_i + \boldsymbol{\mu}_d) - \mathbf{E} \cdot (\mathbf{J}_i + \mathbf{J}_c + \mathbf{J}_d)$$

or alternatively,

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \cdot (\boldsymbol{\mu}_i + \boldsymbol{\mu}_d) - \mathbf{E} \cdot (\mathbf{J}_i + \mathbf{J}_c + \mathbf{J}_d)$$

Energy and Power (Cont'd)

- Integrating over the volume of interest

$$\iiint_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) dV = -\iiint_V \mathbf{H} \cdot (\boldsymbol{\mu}_i + \boldsymbol{\mu}_d) dV - \iiint_V \mathbf{E} \cdot (\mathbf{J}_i + \mathbf{J}_c + \mathbf{J}_d) dV$$

- Utilizing the divergence theorem, we get

$$\oiint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} + \iiint_V \mathbf{H} \cdot (\boldsymbol{\mu}_i + \boldsymbol{\mu}_d) dV + \iiint_V \mathbf{E} \cdot (\mathbf{J}_i + \mathbf{J}_c + \mathbf{J}_d) dV = 0$$

- Explanation of different terms

$\mathbf{P} = \mathbf{E} \times \mathbf{H}$ is the Poynting vector (W/m^2)

$P_o = \oiint_S \mathbf{P} \cdot d\mathbf{S}$ is the power flowing out of the surface S

$P_S = -\iiint_V (\mathbf{H} \cdot \boldsymbol{\mu}_i + \mathbf{E} \cdot \mathbf{J}_i) dV$ is the supplied power (W)

Energy and Power (Cont'd)

$$P_d = \iiint_V \mathbf{E} \cdot \mathbf{J}_c dV = \iiint_V \sigma \mathbf{E} \cdot \mathbf{E} dV = \iiint_V \sigma |\mathbf{E}|^2 dV$$

=dissipated power (W)

$$P_m = \iiint_V \mathbf{H} \cdot \boldsymbol{\mu}_d dV = \iiint_V \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} dV = \iiint_V \mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} dV$$

$$P_m = \frac{\partial}{\partial t} \iiint_V \frac{1}{2} \mu |\mathbf{H}|^2 dV = \frac{\partial}{\partial t} W_m = \text{magnetic power}$$

W_m = magnetic energy

$$P_e = \iiint_V \mathbf{E} \cdot \mathbf{J}_d dV = \iiint_V \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} dV = \iiint_V \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} dV$$

$$P_e = \frac{\partial}{\partial t} \iiint_V \frac{1}{2} \epsilon |\mathbf{E}|^2 dV = \frac{\partial}{\partial t} W_e = \text{electric power}$$

W_e = electric energy

$$P_s = P_o + P_d + \partial(W_e + W_m) / \partial t \quad \text{conservation of EM energy}$$